The (square)-root of the 2/3 labor share

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Abstract

This paper views the aggregate production function as an emerging property of the dynamic evolution of individual firms' labor– and capital productivities. The shape of the aggregate production at any given time is governed by the distribution of underlying firm productivities. I give the class of firm productivities that result in a Cobb-Douglas aggregate production function and present a dynamic model in which this class of firm productivies arises endogenoulsy as the consequence of idiosyncratic shocks to underlying firm productivity. The model generates tractable predictions for firm size and capital intensity. On the balanced growth path the labor share is generally higher than 1/2 and for a broad set of parameters it equals 64%. In particular, a uniform increase in automation intensity does not change the labor share.

1 Introduction

One of the original Kaldor facts is the relative stability of factor shares (Kaldor, 1961), with a labor share in many countries of around 2/3 (Gollin, 2002), though with a pervasive decline in recent decades (Karabarbounis and Neiman, 2014). Several theories predict a constant labor share, and much recent work studies its decline, but there is little understanding of why it should have been around 2/3 in the first place. This paper provides such a theory: firms face idiosyncratic shocks to labor and capital productivity, unsuccessful firms exit, and new firms enter. On a balanced growth path the model admits an (approximate) aggregate Cobb-Douglas production function with parameters that are functions of the underlying productivity process rather than primitive aggregate exponents.

The key asymmetry is that labor is exogenously scarce whereas capital is reproducible. As capital accumulates, labor costs rise and continuously push the whole labor share distribution

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above 1/2. I decompose growth into three components: labor-saving, capital-saving and the entry of new firms. If capital-saving technology is a relatively minor contributor to overall growth, labor's share of factor costs is $1/\sqrt{2} \approx 0.71$, regardless of growth rate, pace of labor-saving technological growth, or the size of shocks to firm productivities. With average markups of around 10 percent, this gives an aggregate labor share of $0.9/\sqrt{2} \approx 64\%$. The paper therefore links the aggregate production function to the productivity distribution, and links the productivity distribution to the evolution of individual firm productivity. It also delivers firm-level predictions for labor shares and capital / labor ratios. Although the baseline focuses on the Cobb-Douglas aggregate production function, I show how a different evolution of firm productivities delivers an aggregate CES with an elasticity between capital and labor of 1/2.

The paper is split into two distinct parts. First, in Section 2 I present a static model in which the market equilibrium can be represented as an aggregate Cobb-Douglas production function. A final good producer aggregates individual products into a final good. Each product is produced by a firm with labor- and capital-augmenting productivities (a, b). Firms face fixed operating costs and are active only when profitable. I characterize a class of distributions over (a, b) for which the economy aggregates to Cobb-Douglas.

This class of distributions includes, but is not restricted to, independent Pareto distributions. I also show how the distribution of (a, b) must differ to generate an aggregate CES with an elasticity of substitution below one. The static model delivers cross-sectional predictions in addition to aggregate factor shares. In particular, it predicts a beta distribution for firm labor shares and a beta-logistic distribution for the log capital / labor ratio. The parameters of both distributions are tightly linked to the aggregate factor shares.

Naturally, the static result raises the question of why the productivity distribution would belong to this class. In the second part of the paper, I present a dynamic model that endogenizes the active productivity distribution. Firms have Leontief production; labor- and capital-augmenting productivities follow exogenous independent idiosyncratic stochastic processes; incumbent firms exit when continuation value falls to zero; and entrants imitate incumbents imperfectly. On the balanced growth path, the model replicates the static model, and I derive the Cobb-Douglas parameters as functions of the underlying productivity process and of firm churn through exit and entry. The analytical results are exact only for the largest firms, but simulations show that they provide a close approximation to the whole labor-share distribution and therefore to the aggregate labor share. The crucial asymmetry is that capital grows with the economy whereas labor does not. Rising labor costs therefore reshape the active distribution, even though individual labor shares also move with idiosyncratic technology shocks. When capital-augmenting (capital-saving) technology is a small contributor to aggregate growth, labor's share of factor costs is $1/\sqrt{2}$. With an exogenously imposed markup of 10% this delivers a labor share out of GDP of 64%. In extensions, I allow for correlated shocks and CES production at the firm level and characterize how each affects the labor share.

Related literature The large literature on the stability of the labor share can broadly be grouped into two explanations. In one, the aggregate production function is Cobb-Douglas and the labor share is pinned down by the exponents. In the other, technological change takes the form of labor-augmenting improvements (Uzawa, 1961;Acemoglu, 2003). The present model delivers an aggregate Cobb-Douglas production function, but the labor-augmenting component is endogenous to selection. As wages rise, firms with low labor productivity are forced to exit. This exit margin changes the active productivity distribution, compresses the labor-share distribution, and generates a labor share above 1/2.

The static part of the model relates most closely to Houthakker (1955), who derives a decreasing-returns-to-scale aggregate Cobb-Douglas production function in a competitive economy. In his model, individual firms have Leontief technologies, labor and capital productivities are independently Pareto distributed, and firms produce either one unit or zero; only the extensive margin is active. The present paper uses a related aggregation logic but changes three elements. First, in addition to the extensive margin, firms operate on an intensive margin. Second, the productivity distribution belongs to a broader class than independent Pareto distributions. Third, although the baseline model has constant returns to scale, extensions generate either decreasing or increasing returns to scale in the aggregate production function. Levhari (1968) builds on Houthakker (1955) by considering productivity distributions that generate aggregate CES production, and Sato (1969) generalizes firm production beyond Leontief.[1] I likewise allow for firm production beyond Leontief and explicitly give a productivity distribution that generates an aggregate elasticity of substitution below one.

More recently, Jones (2005) considers a competitive economy that selects a single production technology from a Pareto-distributed portfolio of technologies in labor and capital productivity. As factor prices change, the economy moves along this technology frontier in a way that can be summarized by an aggregate production function. In a series of papers, Growiec (2008a, 2008b, 2018) further develops this research program, including by showing

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[1] Sato (1975) provides an elegant book-length treatment of the Houthakker--Levhari model and several extensions. Relatedly, Martinez (2025) studies a putty-clay automation model in which firms differ by a scalar automation type embodied in installed capital. His tractable static example also admits an aggregate CES representation with elasticity below one, but through a one-dimensional automation distribution and a wage-setting cutoff rather than through the two-dimensional log-productivity geometry studied here.

how aggregate CES production can arise. The present paper uses a related but different mechanism. Rather than selecting one point on a technology frontier, the market allocation uses a cross-section of active firm technologies. An advantage of this approach is that it also delivers empirical predictions for firm size, labor shares, and capital intensity. I relate the present model more fully to Jones and Growiec's work below.

The dynamic part of the model is most closely related to Luttmer (2007), who studies an economy in which labor is the only factor of production, firms face idiosyncratic productivity shocks, unproductive firms shut down, and new firms enter. He shows that such a model generates empirically plausible size distributions; in particular, when entry costs are high, the size distribution is Pareto with a tail exponent slightly above one.[2] The present model can be seen as a two-dimensional extension of his: firms face productivity shocks to both capital – and labor productivity. In addition to a stationary normalized size distribution, it therefore generates a distribution of labor shares and capital intensity. Importantly, whereas Luttmer's Pareto result applies to the largest firms, I show in simulations that the analytical results provide a close approximation to the whole labor-share distribution and therefore to the aggregate labor share.[3]

Finally, the paper relates to the empirical literature on the constancy of factor shares and on the elasticity of substitution. Gollin (2002) finds that the labor share around 1990 was between 65% and 80% in most countries. Direct estimates of the elasticity of substitution between capital and labor typically find values below one, with material variation across samples and identification strategies (Chirinko, 2008; Antràs, 2004) and Karabarbounis and Neiman (2014) a notable exception. Oberfield and Raval (2021) is closest in spirit to the present model. They use the empirical distribution of firm labor shares in US manufacturing to obtain an elasticity of substitution of around $0.5 - 0.7$, whereas I derive the distribution of firm labor shares from the productivity distribution characterized in the static model and endogenized in the dynamic model. The main role of the model's Cobb-Douglas case is as a baseline. I show how alternative firm productivity distributions and productivity growth processes can generate an aggregate elasticity of substitution of less than 1.

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[2] Gabaix (1999) presents a model where tail city size is also Pareto distributed with a coefficient above but close to one.

[3] Jones (2005) also endogenizes the distribution of productivity, but through a sequence of discrete draws from a Pareto distribution, which is distinct from the mechanism employed here. Lagos (2006) also derives a Cobb-Douglas aggregate from a Pareto distribution of active production units in a search-and-matching model, and shows how this distribution can arise from shocks, job creation, and job destruction.

2 The Static Model

I present the baseline static model, which consists of a final good aggregator which aggregates individual products, produced by firms which differ in labor – and capital productivities. The novel part is the distribution of these firm productivities.

2.1 The Final Good Sector

A final-good sector produces output according to a CES production function:

$$Y = \left[\int_{i \in \Omega} y_i^{\frac{\nu-1}{\nu}} di \right]^{\frac{\nu}{\nu-1}}, \qquad \nu > 1,$$

where each product y_i is produced by an individual firm and sold at the price p_i, and the active set of products (and firms) is given by Ω. Output is used for consumption and the fixed costs of operating firms as described below.

Given prices $\{p_i\}_{i \in \Omega}$, the demand for each individual product is given by

$$y_i = (p_i/P)^{-\nu} Y, \qquad P = \left[\int_{i \in \Omega} p^{1-\nu} di \right]^{\frac{1}{1-\nu}},$$

where the price index, P, has not yet been normalized to 1.

2.2 Firms

I first describe the production technology, profits and production schedules for individual firms as a function of their underlying productivities. I then restrict attention to a particular class of productivity distributions.

2.2.1 Production technology and the zero-profit cutoff

Each product y_i is supplied by a unique producer using the production function:

$$y_i = f(a_i l, b_i k), \tag{1}$$

where f has constant returns to scale. Let w and R be the prices of labor and capital. The implied unit cost function, c, and the labor share, η, are given by

$$c(w/a_i, R/b_i), \qquad \eta \left(\frac{w/a_i}{R/b_i} \right) = \frac{\frac{w}{a_i} c_1 (w/a_i, R/b_i)}{c(w/a_i, R/b_i)},$$

where w and R are the factor prices of labor and capital, which are identical across firms, c_1 is the derivative of c with respect to its first argument, and η depends only on the ratio of $(w/a)\,/\,(R/b)$ since c_1 is homogeneous of degree 0.

The individual production functions satisfy:

Assumption 1. *The individual production function is continuous, increasing in both arguments, concave, and homogeneous of degree 1. Both inputs are essential* $f(0, bk) = f(al, 0) = 0$ *and it satisfies*

$$\lim_{x\to\infty} f(x,1) < \infty \qquad lim_{x\to\infty} f(1,x) < \infty$$

This assumption is satisfied for a CES production function with an elasticity of substitution strictly less than 1, and guarantees that unit costs remain strictly positive when either $a \to \infty$ or $b \to \infty$: $c(0, R/b_i) > 0$ and $c(w/a_i, 0) > 0$. It is slightly stronger than the assumption that both inputs are essential. Specifically it does not allow individual firms to have Cobb-Douglas production functions. Below, I will occasionally assume that capital and labor are complements such that $\eta\left(\frac{w/a}{R/b}\right)$ is always increasing.

Firms face a fixed operating cost equal to FY, such that each firm pays PFY in fixed cost. The assumption that fixed costs scale with output is mostly made for analytical convenience and guarantees that the aggregate production function is CRS. Below, I show that loosening this assumption yields a non-CRS Cobb-Douglas function.

Each firm faces a constant demand elasticity, charges the price $p_i = \frac{\nu}{\nu-1}c(w/a_i, R/b_i)$, and earns profits:

$$\pi = \left\{\kappa_\nu c(w/a_i, R/b_i)^{1-\nu} P^{\nu-1} - F\right\} PY, \tag{2}$$

where $\kappa_\nu \equiv \nu^{-\nu}(\nu-1)^{\nu+1}$.

Equation (2) gives the highest cost at which production is profitable at $P\left(F/\kappa_\nu\right)^{\frac{1}{1-\nu}}$, decreasing in F. Firms with costs below this cutoff operate, while firms with costs above it do not. The set of active producers is given by

$$\Omega(w,R) = \{(a,b) : c(w/a, R/b) \leq P(F/\kappa_\nu)^{\frac{1}{1-\nu}}\}.$$

Since both labor and capital are necessary for production, for each a there exists a minimum required $b(a; w, R, P)$ to ensure weakly positive profits. It is given by $c\left(\frac{w}{a}, \frac{R}{b_0(a;w,R,P)}\right) = P(F/\kappa_\nu)^{\frac{1}{1-\nu}}$, where $b_0(a; w, R, P)$ is decreasing in a: higher labor productivity lowers the capital-productivity requirement for weakly positive profits. I further define $c(w/a_0(w,P), 0) = P\left(F/\kappa_\nu\right)^{\frac{1}{1-\nu}}$ as the lowest labor productivity consistent with weakly positive profits when $b \to \infty$.

Define cost-adjusted factor productivities as $\tilde{a} = \frac{a}{w/P}$ and $\tilde{b} = \frac{b}{R/P}$ and note that

$$c\left(\frac{1}{\tilde{a}}, \frac{1}{\tilde{b}_0(\tilde{a})}\right) = (F/\kappa_\nu)^{\frac{1}{1-\nu}},$$

where $\tilde{b}_0(\tilde{a})$ is the productivity cutoff in cost-adjusted productivities and$\tilde{a}_0$ is defined by $c(1/\tilde{a}_0, 0) = (F/\kappa_\nu)^{\frac{1}{1-\nu}}$. Thus, for cost-adjusted productivities, cutoffs are independent of (w, R, P).

2.2.2 The density of labor and capital productivity

The novel part of the model is the distribution of technologies, (a, b). The density is given by

Assumption 2. *Labor - and capital augmenting technologies* $(a, b) \in [a_{min}, \infty) \times [b_{min} \times \infty)$ *are distributed according to.*

$$g(a, b) = \frac{1}{ab}\psi(a^\alpha b^\beta), \qquad a \geq a_{min}, b \geq b_{min}, \tag{3}$$

where $a_{min}, b_{min}, \alpha, \beta > 0$. *The function* $\psi()$ *is continuous and differentiable and scales to ensure that* $g(a, b)$ *is a distribution. It need not be monotonic.*

Further, there exists a $q > \frac{\nu-1}{\alpha+\beta}$ *such that*

$$\psi(x) = O(x^{-q}). \tag{4}$$

Independent Pareto of (a, b) *arises when* $\psi(x) = \alpha\beta a_{min}^\alpha b_{min}^\beta x^{-1}$ *such that* $g(a, b) = \alpha\beta a^{-(\alpha+1)} b^{-(\beta+1)} a_{min}^\alpha b_{min}^\beta$. *In this case the tail requirement is that* $\alpha + \beta > \nu - 1$.

Assumption 2 implies a particular density in $(loga, logb)$ which allows the model to generate an aggregate elasticity of substitution between capital and labor of 1. Rewrite the density in terms of $loga$ and $logb$:

$$\hat{g}(loga, logb) \propto \psi\left(e^{\alpha loga + \beta logb}\right). \tag{5}$$

This implies constant density along straight lines with slope $-\alpha/\beta$ in $(loga, logb)$ space. A useful special case, is $\psi(x) \propto x^{-\tau}$ in which case the density is:

$$\hat{g}(loga, logb) \propto e^{-\tau(\alpha loga + \beta logb)}, \tag{6}$$

where $\tau = 1$ is independent Pareto. Consequently, independent Pareto implies two restrictions: i) straight iso-density curves in $(\log(a), \log(b))$ with slope $-\alpha/\beta$ and ii) exponential decay in density as one moves northwest through iso-density curves. The class of distributions implied by $\psi(x) \propto x^{-\tau}$ keep straight iso-density curves, and exponential decay but separates the rate of decay from α and β. The general class of densities $\psi()$ only restrict to linear iso-density curves. The covariance between $loga$ and $logb$ is allowed to be both positive and negative, unlike the independent Pareto.

A useful parallel is to a production function of the form $y = h(x_1^{\alpha} x_2^{\beta})$ – with Cobb-Douglas as a special case of $h(x) = x$ – for which the iso-quant in $(logx_1, logx_2)$ has the constant slope: $dlogx_2/dlogx_1 = -(\alpha/\beta)$ such that the TRS in relative terms is constant: the required relative increase in x_2 for a given relative decline in x_1 is constant regardless of the production schedule. An economy satisfying Assumption 2 has a similar restriction: When moving along the $loga$ line, the reduction in $logb$ required for the same density of firms is given by α/β. Thus as the production function $h(x_1^{\alpha} x_2^{\beta})$ features a constant elasticity of substitution between inputs along iso-quants, the density implied by Assumption 2 implies a constant elasticity of substitution between capital and labor productivity as the economy moves along lines with constant firm density.

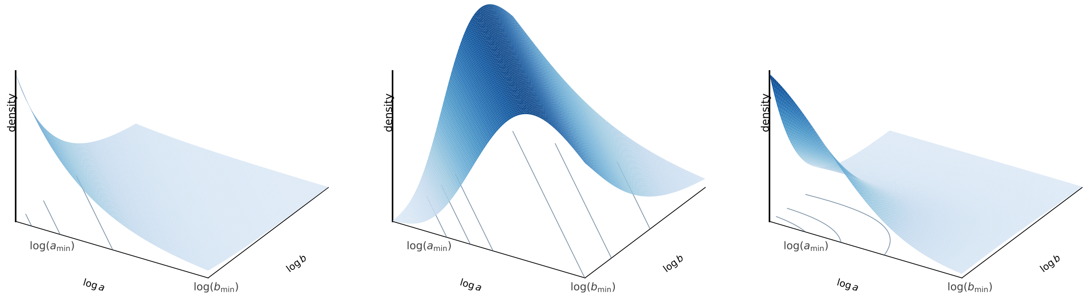
I illustrate three cases of densities $\hat{g}(loga, logb)$, all for $\alpha = \beta$, in Figure 1. Panel a shows the independent Pareto distribution which has constant density along $\beta logb + \alpha loga = k$ and exponential decay at the rate $\alpha + \beta$ as k increases. Panel b shows a distribution with $\psi(z) \propto z^{-\tau} e^{-\lambda/z}$ which also has constant density along linear rays but takes a non-monotonic shape and decays asymptotically at $(\alpha + \beta)\tau$ as k increases. Finally, panel c shows an assumption that does not satisfy (3): $g_{log}(loga, logb) \propto exp\left(\frac{loga+logb}{2}\right)\left(\lambda_L^2 e^{loga} + \lambda_K^2 e^{logb}\right)^{-(\alpha+\beta+1)}$ for $\lambda_L, \lambda_K > 0$. For this distribution the linear rays do not have constant density, but instead take a maximum in the interior. In Section 2.5.3 below, I show that the density of panel c admits an aggregate CES production function with an elasticity of substitution of 1/2.

The restriction on the tail of $\psi(x)$ is to ensure finite production. In particular for a Pareto distribution it requires $\alpha + \beta > \nu - 1$, such that productivity declines fast enough compared to the substitutability parameter ν.

2.3 Market Equilibrium

Only firms demand labor and capital directly. The economy has an exogenous stock L of labor and an exogenous stock K of capital. Each firm's demand for labor and capital follows from Shephard's lemma:

$$wl_i = \eta_i c_i y_i \qquad Rk_i = (1 - \eta_i) c_i y_i$$



(a) Independent Pareto (b) Non-monotonic, Pareto tail (c) Non-constant along rays

Figure 1: Possible distributions of $\hat{g}(loga, logb)$

Notes: Panel (a) uses $\psi(z) \propto z^{-1}$. Panel (b) uses $\psi(z) \propto z^{-\tau}e^{-\lambda/z}$. Panel (c) lets density vary along $\alpha loga + \beta logb = u$. Each panel shows the density of $\hat{g}(loga, logb)$ for different assumptions on ψ. They all illustrate lines along the plane where density is constant. For panel (a) and (b) the iso-density lines are straight, for panel (c) they are not.

such that market clearing for labor and capital requires:

$$wL = \left(\frac{\nu}{\nu-1}\right)^{-\nu} \int_{max\{a_{min}, a_0(w,R)\}}^{\infty} \int_{max\{b_{min}, b_0(a;w,R)\}}^{\infty} \eta(w/a, R/b) c(w/a, R/b)^{1-\nu} P^{\nu} g(a,b) dbdaY, \tag{7}$$

$$RK = \left(\frac{\nu}{\nu-1}\right)^{-\nu} \int_{max\{a_{min}, a_0(w,R)\}}^{\infty} \int_{max\{b_{min}, b_0(a;w,R)\}}^{\infty} \left(1 - \eta(w/a, R/b)\right) c(w/a, R/b)^{1-\nu} P^{\nu} g(a,b) dbdaY, \tag{8}$$

where the lower limits of the integral is determined either by the lower productivity of or the zero profit condition. In what follows, I will impose that the zero-profit cutoff determines the lower integration limits:

Assumption 3. *The equilibrium is on the interior-cutoff branch. The lower support bounds are slack relative to the zero-profit frontier:*

$$a_{\min} < a_0(w, R, P) \qquad \textit{and} \qquad b_{\min} < \lim_{a\to\infty} b_0(a; w, R, P).$$

Thus the zero-profit cutoff, rather than the exogenous lower support, determines the lower integration limits.[4]

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[4] A fully primitive version of this assumption is difficult in general because the cutoff depends on equilibrium real factor prices and therefore on aggregate factor scarcity. In the special case with independent Pareto productivities and Leontief production, the condition reduces to an interval restriction on the factor-endowment ratio:

$$D_L^{1/\beta} \frac{a_{\min}}{b_{\min}} < \frac{\alpha K}{\beta L} < D_L^{-1/\alpha} \frac{a_{\min}}{b_{\min}},$$

Under this assumption, $\max\{a_{\min}, a_0(w, R, P)\} = a_0(w, R, P)$ and likewise for b_0. This guarantees that the extensive margin is active along both the a and b dimensions. The assumption selects the interior-cutoff branch of the static model: the active set is bounded by the zero-profit condition, not by an exogenous lower support bound on productivities. If the active set of firms is instead determined by the support of the productivity distribution, changes in relative factor prices alter not only the location of the zero-profit frontier but also which cost-share directions are available at the margin. This introduces a composition effect and generally breaks the exact Cobb-Douglas aggregation. The aggregate elasticity of substitution is then smaller than 1 because a margin of adjustment is restricted. Appendix 6.2 further develops this argument and shows that if the Pareto coefficient of the revenue distribution is close to 1 the aggregate elasticity of substitution remains close to 1. In the dynamic model below, the productivity distribution endogenously extends to the shutoff condition.

Imposing Assumption 3, equations (7) and (8), together with the price index,

$$P = \frac{\nu}{\nu - 1}\left[\int_{a_0(w,R)}^{\infty}\int_{b_0(a;w,R)}^{\infty} c(w/a, R/b)^{1-\nu} g(a,b)\,db\,da\right]^{\frac{1}{1-\nu}}, \tag{9}$$

pin down $(w/P, R/P, Y)$ in market equilibrium. The equilibrium exists and is unique under the conditions of Assumption 2.

Changes in factor prices affect factor demand through three channels: changes in the extensive margin, summarized by (a_0, b_0); factor substitution within individual firms, summarized by η; and the rescaling of firm production induced by relative cost changes. To separate these effects, it is useful first to rewrite the price index in terms of the cost-adjusted productivities, $\tilde{a} = a/(w/P)$ and $\tilde{b} = b/(R/P)$ and note that the price index in (9) becomes:

$$1 = \frac{\nu}{\nu - 1}P\left[\int_{\tilde{a}_0}^{\infty}\int_{\tilde{b}_0}^{\infty} c(1/\tilde{a}, 1/\tilde{b})^{1-\nu}\psi(\tilde{a}^{\alpha}\tilde{b}^{\beta}w^{\alpha}R^{\beta}P^{-(\alpha+\beta)})\,d\tilde{b}\,d\tilde{a}\right]^{\frac{1}{1-\nu}},$$

which pins down a unique $P = \kappa_P w^{\frac{\alpha}{\alpha+\beta}} R^{\frac{\beta}{\alpha+\beta}}$. The same substitution for the labor-market clearing condition gives:

$$\frac{wL}{PY} = \left(\frac{\nu}{\nu - 1}\right)^{-\nu}\int_{\tilde{a}_0}^{\infty}\int_{\tilde{b}_0(\tilde{a})}^{\infty} \eta(1/\tilde{a}, 1/\tilde{b})c(1/\tilde{a}, 1/\tilde{b})^{1-\nu}\psi(\tilde{a}^{\alpha}\tilde{b}^{\beta}\kappa_P^{-1})\,d\tilde{b}\,d\tilde{a} \equiv \kappa_L,$$

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where $D_L = \frac{(\nu-1)^2}{\nu F}\frac{\alpha+\beta-\nu+1}{\alpha\beta B(\alpha,\beta)}$. The interval is nonempty if and only if $D_L < 1$, equivalently the fixed operating cost is large enough relative to the tail parameters.

and $\frac{RK}{PY} = \kappa_K$ analogously. Combining these elements gives:

$$P^{\alpha+\beta} = \kappa_P^{\alpha+\beta} w^\alpha R^\beta = \kappa_P \left(\frac{\kappa_L}{L} PY\right)^\alpha \left(\frac{\kappa_K PY}{K}\right)^\beta \Leftrightarrow$$

$$Y \propto L^{\frac{\alpha}{\alpha+\beta}} K^{\frac{\beta}{\alpha+\beta}},$$

and Shephard's lemma on the aggregate cost function P gives: $\frac{wL}{PY} = \frac{\nu-1}{\nu}\frac{\alpha}{\alpha+\beta}$ and analogously for $RK/(PY) = \frac{\nu-1}{\nu}\frac{\beta}{\alpha+\beta}$. I collect this in Proposition 1

Proposition 1. *The equilibrium exists and is unique. Under Assumption 1 and Assumption 2 (including the integrability tail condition), there exists a unique constant $\kappa_P > 0$ such that the equilibrium price index satisfies $P = \kappa_P w^{\frac{\alpha}{\alpha+\beta}} R^{\frac{\beta}{\alpha+\beta}}$.*

Output is given by:

$$Y \propto L^{\frac{\alpha}{\alpha+\beta}} K^{\frac{\beta}{\alpha+\beta}},$$

Factor payments are given by:

$$\frac{wL}{PY} = \frac{\nu-1}{\nu}\frac{\alpha}{\alpha+\beta}, \qquad \frac{RK}{PY} = \frac{\nu-1}{\nu}\frac{\beta}{\alpha+\beta},$$

with aggregate profits (gross of fixed cost) given by $\frac{1}{\nu}PY$.

To gain some intuition for why the specification in Assumption 2 delivers a price index of $P = \kappa_P w^{\frac{\alpha}{\alpha+\beta}} R^{\frac{\beta}{\alpha+\beta}}$, consider two firms with technologies $(loga^A, logb^A)$ and $(loga^B, logb^B)$ on the same ray $\alpha loga + \beta logb = k$. Let firm B have higher labor productivity, a than firm B by some factor δ: $loga^B = loga^A + \delta \Leftrightarrow a^B = a^A e^\delta$, with $b^B = b^A e^{-\frac{\beta}{\alpha}\delta}$. By assumption the mass at these two points is the same, but firms at each point will have different labor shares and, in general, different costs, and since costs determine whether a firm is active, they might not both be active.

Now consider initial factor costs (w, R) and let the wage increase by δ in logs: $w' = we^\delta$along with a decline in the cost of capital to $R' = Re^{-\frac{\beta}{\alpha}\delta}$ so that the aggregate price index, $P \propto w^{\frac{\alpha}{\alpha+\beta}} R^{\frac{\beta}{\alpha+\beta}}$ is kept constant. Consider the cost of firm B after the price change:

$$c^B_{after} = c\left(\frac{we^\delta}{a^B}, \frac{Re^{-\frac{\beta}{\alpha}\delta}}{b^B}\right) = c\left(\frac{w}{a^B e^{-\delta}}, \frac{R}{b^B e^{\frac{\beta}{\alpha}\delta}}\right) = c\left(\frac{w}{a^A}, \frac{R}{b^A}\right) = c^A_{before},$$

Thus, the cost of firm B after the price change equals the cost of firm A before the price change. Consequently, the economy can "copy" production from A to B following the price change and replicate the same allocation at a higher a/b. Since the density at the two points is the same, this is possible. Because the extensive margin is a function of cost alone, the zero-profit cutoff moves accordingly as the economy transitions along the ray. α/β represents the opportunity cost in terms of b of moving towards a higher a. When $\alpha > \beta$ the a distribution is more compressed and it is costlier for the economy to move towards higher a. Consequently, the price index scales more intensely with w and the exponent on the Cobb-Douglas price index is higher.

I illustrate this in Figure 2 using a single-technology economy in the spirit of Jones (2005). Consider an economy endowed with fixed factors, L and K which can choose from a single active production technology from a portfolio of the form $Y = f(aL, bK)$ restricted to the set $\alpha loga + \beta logb = u$, or equivalently $a^\alpha b^\beta = exp(u)$, a Pareto technological frontier. The function f has constant returns to scale and an elasticity of substitution of less than 1. The technology frontier is illustrated in Panel a as a straight line with slope $-\alpha/\beta$. The slopes of isoquants of $f(aL, bK)$ for given (L, K) are given by:

$$\frac{dlogb}{dloga} = -\frac{f_1 \times aL}{f_2 \times bK}.$$

f has an elasticity of substitution of less than 1, such that an increase in K/L increases the slopes of the isoquants everywhere. In a competitive market economy, the optimum is where the isoquant is tangent to the technological frontier, and factor prices equal marginal products, $w = f_1 a$ and $R = f_2 b$ so the economy operates where:

$$\frac{\alpha}{\beta} = \frac{f_1 aL}{f_2 bK} = \frac{wL}{RK}.$$

Thus the relative factor shares are given by the ratio of the Pareto coefficients. The figure further illustrates the effect of an increase in K/L: the isoquants become steeper everywhere and the economy transitions toward a production function with a higher a and lower b, by utilizing technology with a higher labor productivity and lower capital productivity. It always satisfies $wL/(RK) = \alpha/\beta$ and therefore $wL/(wL + RK) = \alpha/(\alpha + \beta)$. As shown by Jones (2005), aggregate production can be represented by a Cobb-Douglas $Y \propto L^{\frac{\alpha}{\alpha+\beta}} K^{\frac{\beta}{\alpha+\beta}}$. A high α/β implies that the transition towards higher a is costly in terms of b and the gain to production of higher K is muted.

Panel b shows the analogous figure for the multi-firm economy studied here. The plane is $(\log a, \log b)$ and the vertical axis is revenue at a given point scaled by the density at that point. In the plane, the figure shows two iso-density lines $\alpha loga + \beta logb = u$. The figure further shows the zero-profit cutoff in a and b for given (w, R): Firms with costs that are too high are not active. This is illustrated by the discrete cutoff at the end of the revenue distribution

The firm on this segment with the highest production, and therefore revenue, is the one with the lowest cost, given by

$$min_{a,b}c(w/a, R/b)\, s.t.\, \alpha loga + \beta logb = u,$$

whose first-order condition delivers

$$\frac{\eta}{1-\eta} = \frac{\alpha}{\beta},$$

That is, the firm with the same labor share as the overall economy along the ray $\alpha loga + \beta logb = u$ has the highest revenue, with revenue declining on both sides until the point where fixed costs make production unprofitable, at which point there is a discrete decline in revenue. The higher the elasticity of substitution, ν, the more concentrated the figure becomes, until it converges to a single point corresponding to the single production technology.

There is a continuum of such curves, with two such curves illustrated in the figure. The panel demonstrates that this economy extends the single-technology economy in two dimensions: First, instead of choosing one technology along a ray, multiple technologies / firms are active, but the highest revenue occurs under the same condition as in the single-technology economy. Second, instead of choosing a technology along a single ray, the economy has active production along a continuum of rays.

Panel c illustrates the effect of an increase in w/R following a relative increase in K/L. This pushes the whole distribution down along the line as the zero-profit cutoff line moves accordingly. The economy mirrors the single-technology economy. The figure is drawn along a single ray for clarity, but the revenue distribution shifts down along all rays. The set of active firms moves concomitantly, as illustrated by the movement of the zero-profit cutoff on the horizontal plane. The steeper are these iso-density curves, the more difficult the transition towards firms with higher labor technology and the higher is the labor share. As ν and $\alpha + \beta$ tend to infinity, the multi-firm economy converges to the single-technology economy.[5]

Having established the properties of the aggregate production function and factor payments, I now turn to the firm-level distributions of revenue, the labor share (out of factor cost), and capital-labor ratios. A strength of the paper is that it delivers direct testable predictions on these.

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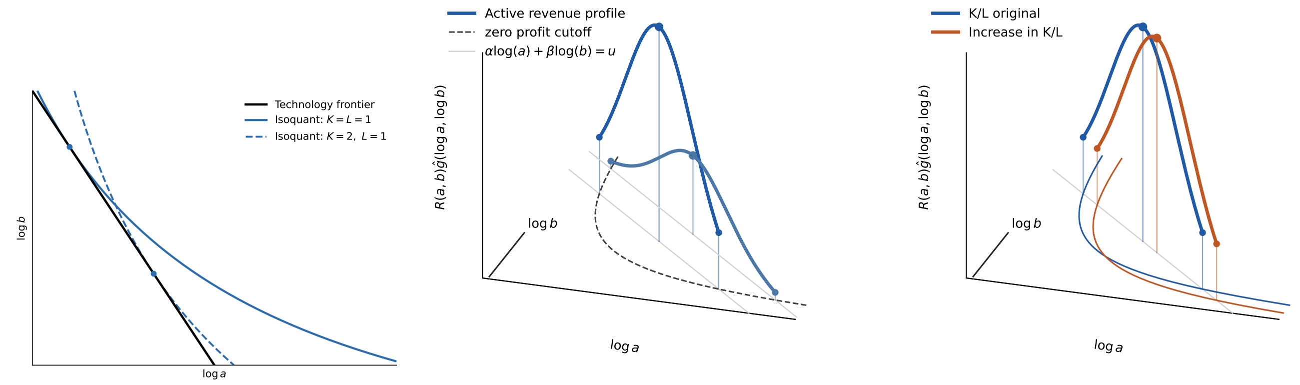
[5] Specifically, for independent Pareto, let $\nu \to \infty$ with α, β increased so that α/β is constant and $\alpha + \beta - \nu + 1 \to \infty$. Then the economy converges to a single ray, and with near-perfect substitutes only one firm (with no market power) is active. It is necessary to increase $\alpha + \beta$ along with ν since g is unbounded and, for given g, taking the limit to perfect substitutes, $\nu \to \infty$, generates infinite output and is violation of the Assumption 2.



(a) Single-technology (b) Multiple products (c) Multiple products: Increase in K/L

Figure 2: Single-technology and multi-firm intuition. Panel (a) shows the single-technology benchmark. Panel (b) shows revenue contributions across firms along several rays with the zero-profit cutoff shown in the horizontal plane. Panel (c) shows how the revenue profile on a given ray shifts after an increase in K/L .

2.4 The distributions of firm size, the labor share and the capital-labor ratio

I now derive the model's implications for the firm-size tail, the labor-share distribution, and the capital-labor distribution. Henceforth, the price level is normalized to $P = 1$. Throughout, I maintain a constant-elasticity-of-substitution production function at the firm level:

$$y = \left[(al)^{\frac{\epsilon-1}{\epsilon}} + (bk)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}},$$

where $0 \leq \epsilon < 1$ is the elasticity of substitution between capital and labor at the firm level.[6]

For the CES production function, the associated unit-cost and labor-share functions are

$$c(w/a, R/b) = \left[(1/\tilde{a})^{1-\epsilon} + (1/\tilde{b})^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}, \qquad \eta = \frac{(1/\tilde{a})^{1-\epsilon}}{(1/\tilde{a})^{1-\epsilon} + (1/\tilde{b})^{1-\epsilon}},$$

where again η is labor share out of cost not of revenue and $\tilde{a} = a/(w/P) = a/w$ and $\tilde{b} = b/R$. Write $(\tilde{a}, \tilde{b})$ as direct functions of (η, c) to get

$$\tilde{a} = c^{-1}\eta^{-\frac{1}{1-\epsilon}} \qquad \tilde{b} = c^{-1}(1-\eta)^{-\frac{1}{1-\epsilon}}. \tag{10}$$

I first consider distributions with power-law decay in this index: $\psi(x) \propto x^{-\tau}$, as in Panel b

---

[6] The production function is symmetric in its two arguments. Any weight would be isomorphic to a rescaling of (a, b) and a rescaling of the distribution a_{min} and b_{min}.

of Figure 1, with independent Pareto when $\tau = 1$. For this class of distributions, the labor-share distribution will be Beta distributed and independent of firm size. The capital-labor ratio follows a beta-logistic distribution. I then extend these distributional results to more general choices of $\psi(x)$.

Power-law decay $\psi(x) \propto x^{-\tau}$ Let the distribution g satisfy: $\psi(x) \propto x^{-\tau}$ such that in $(\tilde{a}, \tilde{b})$ it takes the form:

$$g_{\tilde{a},\tilde{b}}(\tilde{a}, \tilde{b}) \propto \frac{1}{\tilde{a}\tilde{b}} \left(\tilde{a}^{\alpha}\tilde{b}^{\beta}\right)^{-\tau},$$

as above the active-set boundary $(\tilde{a}_0, \tilde{b}_0(\tilde{a}))$ is independent of (w, R) under Assumption 3.

A direct change of variables in the distribution delivers:[7]

$$f_{c,\eta}(c, \eta) \propto \eta^{\frac{\alpha\tau}{1-\epsilon}-1}(1-\eta)^{\frac{\beta\tau}{1-\epsilon}-1}c^{(\alpha+\beta)\tau-1}, \qquad for\, \eta \in (0,1)\ and\, c < (F/\kappa_{\nu})^{\frac{1}{1-\nu}},$$

Thus, the labor share is independent of cost and therefore of firm size. The parameters of the Beta distribution are $(\frac{\alpha\tau}{1-\epsilon}, \frac{\beta\tau}{1-\epsilon})$, the mean is $\alpha/(\alpha+\beta)$ and the mode of the distribution is $\frac{\alpha - \frac{1-\epsilon}{\tau}}{\alpha+\beta-2\frac{1-\epsilon}{\tau}}$, both of which are greater than 1/2 if and only if $\alpha > \beta$. Thus, the mean labor share within each size class mirrors the aggregate labor share out of factor cost, $\alpha/(\alpha+\beta)$, and the mean is constant across the whole size distribution. A higher elasticity of substitution, ϵ, dampens the effect of differences in a/b on labor shares, and consequently the distribution of labor shares is more concentrated, but the mean remains the same.[8]

I collect these distributional implications, together with the firm-size tail and the distribution of $log(k/l)$ in the proposition below.

Proposition 2. *Consider a distribution of (a, b) given by $\psi(x) \propto x^{-\tau}$. The following results hold for the active set of firms.*

---

[7] Replace $(\tilde{a}, \tilde{b})$ with (η, c) in $g_{\tilde{a},\tilde{b}}$ and note that the Jacobian is $\left|\frac{\partial(\tilde{a},\tilde{b})}{\partial(c,\eta)}\right| = \frac{1}{\epsilon-1}c^{-3}\eta^{-\frac{1}{1-\epsilon}-1}(1-\eta)^{-\frac{1}{1-\epsilon}-1}$,

[8] Oberfield and Raval (2021) use the empirical distribution of the labor share in US manufacturing to decompose the aggregate elasticity of substitution into reallocation within firms, reallocation across firms and entry/exit. This model features the same three channel, but relies on the theoretically derived distribution labor share distribution. Let ϵ_{agg} denote the aggregate elasticity of substitution, and consider the independent Pareto distribution. ϵ_{agg} can then be decomposed into:

$$\epsilon_{agg} - 1 = \underbrace{-\frac{(\alpha+\beta)(1-\epsilon)}{\alpha+\beta+1-\epsilon}}_{\text{within firm substitution}(<0)} \underbrace{+\frac{(1-\epsilon)(\nu-1)}{\alpha+\beta+1-\epsilon}}_{\text{Between firm reallocation}(>0)} \underbrace{+\frac{(1-\epsilon)(\alpha+\beta-\nu+1)}{\alpha+\beta+1-\epsilon}}_{\text{Entry / exit}(>0)} = 0.$$

The three terms are within-firm substitution, reallocation across continuing firms, and movement of the zero-profit boundary. Thus, an increase in individual firm elasticity of substitution directly pushes the aggregate elasticity of substitution up, but at the same time, for given productivity distribution, it compresses the equilibrium distribution of labor shares, and dampens the reallocation and entry/exit margin, leaving the aggregate elasticity of substitution unchanged.

a) The revenue share of individual firms, $R_i = p_i y_i / Y$, has a Pareto upper tail:

$$\Pr(R_i > z \,|\, active) \sim \mathcal{C} z^{-\frac{(\alpha+\beta)\tau}{\nu-1}},$$

where $\mathcal{C}$ is a positive constant.

b) The labor share out of cost, $\eta_i = \frac{wl}{wl+Rk}$ is independent of cost and firm size and is Beta-distributed:

$$f_{\eta|c}(\eta|c) \propto \eta^{\left(\frac{\alpha\tau}{1-\epsilon}-1\right)}(1-\eta)^{\left(\frac{\beta\tau}{1-\epsilon}-1\right)}$$

with $\mathbb{E}(\eta_i \mid c) = \mathbb{E}(\eta_i) = \frac{\alpha}{\alpha+\beta}$.

c) The log capital-labor ratio, $\log(k_i/l_i)$, has a beta-logistic distribution, independent of size, with density

$$f_{log(k/l)}(x) \propto e^{\left(\frac{\beta\tau}{1-\epsilon}[x-log(w/R)]\right)} \left(1+e^{x-log(w/R)}\right)^{-\frac{(\alpha+\beta)\tau}{1-\epsilon}}, \qquad x \in \mathbb{R},$$

where $log(w/R)$ affects the scale of the distribution, but not its shape.

Proof. Appendix 6.12. $\square$

The revenue tail has a simple intuition: When both capital and labor are essential, the productivity distribution decays at rate $(\alpha+\beta)$ for Pareto and $\tau(\alpha+\beta)$ for the broader power-law class. Revenue scales with $c^{-\frac{1}{\nu-1}}$ and consequently the revenue tail parameter is $(\alpha+\beta)\,\tau/(\nu-1)$. The distribution of $log(k/l)$ follows directly from the distribution of η. The distribution of η is independent of (w/R). Further, only the scale of the $log(k/l)$ distribution depends on w/R, not its shape, which is determined entirely by α, β and $\tau/(1-\epsilon)$. The left tail of $log(k/l)$ pins down $\beta\tau/(1-\epsilon)$ and the right tail pins down $\alpha\tau/(1-\epsilon)$. Thus, although the level of α and β cannot be determined from the shape of the $log(k/l)$ distribution, the relevant object, $\alpha/(\alpha+\beta)$ is determined by the relative thickness of the left and right tails.

Proposition 2 considered the exact power-law case. The next proposition shows that two useful features survive more generally.

A general class of $\psi(x)$. An analogous proposition can be established under a more general class of $\psi(x)$

Proposition 3. *Consider the balanced class $g(a,b) = \frac{1}{ab}\psi(a^\alpha b^\beta)$ and a CES production function with elasticity of substitution $\epsilon \in [0,1)$.*

a) Under the specification $\psi(z) = Dz^{-\tau}e^{-\lambda/z}$ (Panel b of Figure 1), the conditional distribution of η_i given cost c, and therefore given firm size, is an exponentially tilted Beta *distribution:*

$$f_{\eta|c}(\eta \mid c) \propto \eta^{\frac{\alpha\tau}{1-\epsilon}-1}(1-\eta)^{\frac{\beta\tau}{1-\epsilon}-1}\exp\left\{-\theta c^{\alpha+\beta}\eta^{\frac{\alpha}{1-\epsilon}}(1-\eta)^{\frac{\beta}{1-\epsilon}}\right\}, \qquad 0<\eta<1,$$

with θ a positive constant. $f_{\eta|c}$ converges to beta $\eta^{\frac{\alpha\tau}{1-\epsilon}-1}(1-\eta)^{\frac{\beta\tau}{1-\epsilon}-1}$ as $c \to 0$. Here, θ is a positive constant.

Second, under CES production and for any admissible ψ in the balanced interior class, it holds that $\mathbb{E}(\eta_i \mid c) = \alpha/(\alpha+\beta)$ at every interior cost level. Thus, exact independence of η_i and size is special to the power-law case, but the conditional mean is not. Any distribution that delivers a Pareto firm-size tail also has a Beta-distributed labor share (and beta-logistic distribution for log(K/L) for large firms.

Proof. Appendix 6.13. ☐

Proposition 3 generalizes the labor-share distribution. It shows that the mean labor share remains $\alpha/(\alpha+\beta)$ independent of size throughout the distribution, even though the specific Beta form is not generic. For any class of distributions with a Pareto firm-size tail, the Beta result holds asymptotically, with an analogous result for the distribution of *log*(K/L). This property also appears in the dynamic model below and is a direct firm-level prediction. In particular it ties the labor share out of factor costs tightly to the shape of the *log*(K/L) distribution.

2.5 Extensions and robustness in the static model

This section considers a series of extensions of the model, focusing on whether they break the unitary elasticity between capital and labor. I first allow heterogenous markups and fixed costs of a particular type that maintain unitary aggregate elasticity of substitution between capital and labor as well as constant returns to scale in the aggregate. I then allow for decreasing returns to scale in individual production and fixed cost that do not scale with output. Those maintain the unitary elasticity of substitution but break aggregate constant returns to scale. I then consider extensions that break unitary elasticity: a different class of productivity distributions that break unitary elasticity and a zero profit condition that is not binding.

2.5.1 CRS Cobb-Douglas production functions

I first allow firms to vary by markup or by operating fixed cost. Under particular independence assumptions, the aggregate Cobb-Douglas structure survives.

Heterogeneous markups I allow for heterogeneous markups with a particular structure. Replace the CES production function of the final-good sector with the following Kimball-style aggregator:

$$1 = \int_{\Omega} \Psi\left(y_i/Y, \gamma_i\right) di,$$

where γ_i is specific to firm i and $\Psi(y_i/Y, \gamma_i) = (y_i/Y)^{\frac{\nu-1}{\nu}}$ is the CES special case. That, is this specification allows firms to have intrinsically different demand elasticities, captured by γ_i, and for those demand elasticities to vary with scale (as in the original Kimball paper which specifies the same Ψ for all firms). Certain regularity conditions on Ψ are required to ensure that an equilibrium exists. I state a sufficient set of such conditions in Appendix 6.14. The joint distribution over (a, b, γ) is such that

$$g(a, b \mid \gamma) = \frac{1}{ab}\psi_{\gamma}(a^{\alpha}b^{\beta}),$$

with the same α, β for every γ. This is weaker than independence of γ and (a, b). In Appendix 6.14 I show that when the zero-profit cutoff holds on all rays and for all values ofγ, the economy admits a Cobb-Douglas production function

$$Y \propto L^{\frac{\alpha}{\alpha+\beta}} K^{\frac{\beta}{\alpha+\beta}} \qquad \frac{wL}{pY} = \mathcal{V}_{*}\frac{\alpha}{\alpha+\beta},$$

with $(wL)/(RK) = \alpha/\beta$. Markups will now vary across firms such that $\mathcal{V}_{*}$ depends on the whole distribution of γ but is independent of K, L. In essence, for each γ the set of firms reproduces the model above, with Cobb-Douglas exponents depending only on α, β and integrating over different γ retains the same Cobb-Douglas structure.

Heterogeneous fixed costs An analogous argument applies to heterogeneous fixed costs. Let firms vary by (F, a, b) where fixed costs are still proportional to Y and let $g(a, b|F) = \frac{1}{ab}\psi_F(a^{\alpha}b^{\beta})$. Then the economy admits an aggregate production function $Y \propto L^{\frac{\alpha}{\alpha+\beta}} K^{\frac{\beta}{\alpha+\beta}}$, with $\frac{wL}{PY} = \frac{\nu-1}{\nu}\frac{\alpha}{\alpha+\beta}$. The proof follows the same steps as the proof for heterogeneous markups and is omitted.

2.5.2 Non-CRS Cobb-Douglas production functions

In the following I change the constant returns to scale for individual firms or the proportionality of fixed cost. I show that both preserve the Cobb-Douglas form, but break constant returns to scale.

Non-constant returns to scale for individual firm production Change the individual production functions to

$$\tilde{f}(al, bk) = f(al, bk)^{\vartheta},$$

where $0 < \vartheta \leq 1$, a simple form of decreasing returns to scale. Appendix 6.15 shows that the economy admits a Cobb-Douglas production function:

$$Y \propto L^{\frac{\vartheta\alpha}{\alpha+\beta}} K^{\frac{\vartheta\beta}{\alpha+\beta}},$$

Thus, the aggregate production function inherits the decreasing returns to scale from individual firms. The relative factor returns are still $wL/(RK) = \alpha/\beta$. In this setting profits for firms come both from decreasing returns to scale and the markup.

Fixed costs that are not proportional to output Now replace the operating fixed cost FY with FY^{φ}, where $0 \leq \varphi \leq 1$. $\varphi = 1$ is the case considered in the main text and $\varphi = 0$ corresponds to fixed costs that are independent of Y. Appendix 6.16 shows that, in the interior power-law case ($\psi(x) \propto x^{-\tau}$), the economy admits a Cobb-Douglas production function:

$$Y \propto L^{\frac{\alpha}{\alpha+\beta}p(\varphi)} K^{\frac{\beta}{\alpha+\beta}p(\varphi)}, \qquad p(\varphi) = \frac{(\alpha+\beta)\tau(\nu-1)}{(\alpha+\beta)\tau(\nu-1) - (1-\varphi)\left((\alpha+\beta)\tau + 1 - \nu\right)},$$

Thus, φ introduces increasing return to scale through $p(\varphi)$, with $p(1) = 1$ and $p(\varphi) > 1$ for $\varphi < 1$. The relative factor payments continue to be $wL/(RK) = \alpha/\beta$. For the broader class of functions $\psi(x)$, aggregate production can be written as $F(L, K) = \Gamma(L^{\frac{\alpha}{\alpha+\beta}} K^{\frac{\beta}{\alpha+\beta}})$, with $\Gamma' > 0$ but $wL/(RK) = \alpha/\beta$.

2.5.3 An aggregate CES function with elasticity of substitution 1/2

The Cobb-Douglas case above is not mechanical. It uses the fact that the density of firm types leaves enough mass along the directions in log-productivity space that change relative factor bias. I now give a small extension in which this property fails and the aggregate elasticity of substitution is $\varepsilon = 1/2$.

Let individual firms have Leontief unit costs,

$$c(a, b; w, R) = \frac{w}{a} + \frac{R}{b}.$$

Consider the high-productivity density

$$g(a,b) \propto \frac{a^{-1/2}b^{-1/2}}{(\lambda_L^2 a + \lambda_K^2 b)^{m+1}}, \qquad a \geq a_{\min}, \quad b \geq b_{\min}, \quad m > \nu - 1. \tag{11}$$

The mechanism is variation in mass along the factor-bias direction. Along a line with $\log a + \log b$ fixed, the density in (11) is maximized at $a/b = (\lambda_K/\lambda_L)^2$ and falls as the firm type becomes more unbalanced away from that point. Hence a change in relative factor prices that shifts production toward more factor-biased firms also moves the economy into thinner parts of the type distribution. This breaks the Cobb-Douglas replication argument and makes the aggregate substitution margin less elastic.

Appendix 6.11 derives the factor-demand relation

$$\frac{L}{K} = \frac{\lambda_L}{\lambda_K}\left(\frac{w}{R}\right)^{-1/2}.$$

The aggregate economy therefore behaves as if it had a CES unit cost with elasticity of substitution 1/2. The parameters λ_L and λ_K determine both the relative CES weights and the location of the density peak. The broader CES construction is left to Appendix 6.17.

This is related to the CES aggregation result in Growiec (2008a,b, 2018), but here the relevant object is not a single curved technology frontier. It is how probability mass changes along the factor-bias direction of a two-dimensional type distribution.

2.6 Taking stock

The static model shows that for a certain class of distributions with linear iso-density curves the aggregate economy admits a Cobb-Douglas. When the distribution has exponential decay the model delivers predictions for the size distribution, the labor share distribution and the capital ratio distribution parameterized by α and β. The model predicts that any distribution in (a,b) within this class that has a Pareto tail in the size distribution will have a beta distribution in labor shares and a beta-logistic in the capital ratio. These distributional predictions provide natural targets for future work.

This section has taken the form of $\psi(x)$ as given. The next section derives it from first principle as the result of idiosyncratic productivity increments for individual firms. This (approximately) recreates the functional form of the static model and solves for α and β as a function of the parameters of the technological processes.

3 Dynamic model

The static model takes as given that density is constant along rays $\alpha loga + \beta logb = u$. The dynamic model endogenizes this structure through firm-level movements in technologies a and b. In doing so it endogenizes α and β as functions of primitive parameters. Although the model allows for different technological trends in a and b, the only fundamental asymmetry between capital and labor is that labor is "scarce" in the sense that it is supplied by an exogenous stock, whereas capital is produced in the economy. I show that this scarcity of labor by itself pushes $\alpha > \beta$ and therefore raises the labor share (out of factor cost) above 1/2. The results are exact only for the largest firms. However, the simulations show that the relevant objects, and in particular the labor share, closely approximate the overall distribution[9]. The model is closely related to the model of Luttmer (2007) and approximately replicates its implications for the size distribution.

I proceed in four steps. First, I describe the dynamic process governing firm technologies, profits, exit, and entry, and I characterize the joint dynamics of cost and the labor share for individual firms. Second, I close the model and show that it admits a balanced growth path on which wages, capital, and output grow at the same rate, while the cost of capital is constant. The tail of the size distribution satisfies explicit closed-form conditions, and the tail labor share is pinned down by exogenous parameters. Third, I show computationally that the tail expressions hold to a high degree of approximation throughout the distribution, not just in the tail. Fourth, I consider several extensions.

For ease of exposition, the baseline model assumes Leontief production for individual firms, zero correlation between productivity shocks, and no exogenous firm death. I show that none of these simplifications affect the overall Cobb-Douglas structure of the economy, though they do affect the labor share on the balanced growth path.

3.1 The macroeconomic environment

A representative household owns all labor, physical capital, and claims on firms. Its preferences are

$$U = \int_0^\infty e^{-\rho t} \log C_t \, dt,$$

where $\rho > 0$ is the pure rate of time preference and C_t is aggregate consumption.

As in the static model, a final good is produced competitively. Suppressing time sub-

---

[9] This is contrary to a literature on the size distribution of firms, where the Pareto fit holds only in the tail.

scripts where no confusion arises, output Y and the price level P are given by:

$$Y = \left[\int_{i\in\Omega} y_i^{\frac{\nu-1}{\nu}}\, di\right]^{\frac{\nu}{\nu-1}} 1 \qquad 1 = P = \left[\int_{i\in\Omega} p_i^{1-\nu} di\right]^{\frac{1}{1-\nu}}$$

where Ω is the active set of firms, $\nu > 1$, and the price level is normalized to 1. The normalization of P is important. On the balanced growth path the mass of active firms is constant and the price normalization will imply a stationary cost distribution. With uniform markups it must satisfy: $\left(\frac{\nu-1}{\nu}\right)^{\frac{1}{1-\nu}} = \int_{i\in\Omega} c_i^{1-\nu} di$.

A competitive sector uses the final good to produce capital K_t according to

$$\dot{K}_t = I_t - \delta K_t,$$

where $\delta > 0$ is the depreciation rate. Capital is homogeneous, not subject to adjustment costs, and is rented competitively at rate R. Zero profit in the capital-producing sector and the household Euler equation imply

$$R = r + \delta, \qquad g = r - \rho,$$

where r is the interest rate and g is the common growth rate of consumption and output on the balanced growth path. On the balanced growth path, the rental rate $R = \rho + g + \delta$ is constant. Thus, R plays the role of r in the static model.

3.2 Firm profit maximization and stochastic development

At every given instant firms solve a static maximization problem, and for a given productivity distribution this gives prices, production schedules, and profits. I solve this problem first. After this, I solve for firms only dynamic problem; the question of whether to shut down or to continue producing.

3.2.1 Static firm problem

Each variety i is produced by a unique firm using the Leontief technology

$$f_i(l,k) = \min\{a_i l, b_i k\},$$

where a_i and b_i are firm-specific labor- and capital-augmenting productivities.[10] With wage w and rental rate R, firms unit cost and labor share out of cost are:

$$c(w/a, R/b) = w/a + R/b \qquad \eta = \frac{w/a}{w/a + r/b}, \tag{12}$$

and the firm charges $p = \nu/(1-\nu)c$. Firms continue to pay a fixed cost FY to operate. Define $\chi = log(c)$ as log cost and note that profits are:

$$\Pi Y = \left\{ \kappa_\nu c(w/a_i, r/b_i)^{1-\nu} - F \right\} Y, \tag{13}$$

Let $\chi_0 = \frac{1}{1-\nu} log(F/\kappa_\nu)$ denote the zero profit cost, and write normalized profits, Π as:

$$\Pi(\chi)/F = e^{(1-\nu)(\chi-\chi_0)} - 1. \tag{14}$$

Due to the option value of continuing firms will in general not shut down when they reach χ_0, and unlike the static model, the set of active firms and the set of firms with positive profits are not identical.

3.2.2 Firm productivity development

The productivity of each firm follows geometric Brownian motions such that:

$$dloga = \mu_a dt + \sigma dW^a, \tag{15}$$

$$dlogb = \mu_b dt + \sigma dW^b,$$

where dW^a and dW^b are independent Brownian motions, μ_a and μ_b are deterministic technological trends for incumbent firms and $\sigma > 0$ are the shocks around this trend. μ_a and μ_b are naturally thought of as weakly positive, but need not be.

Equation (15) can be seen as a "productivity"-version of Gibrat's law that mean proportional growth is independent of current size. Movements in cost and labor share follow directly from (12) and (15). On a balanced growth path where w grows at the rate g and R is constant they follow.

Lemma 4. *Suppose that, along the balanced growth path, $d\log w = g\,dt$ and $d\log R = 0$. Let $\chi = \log c$, and $\eta = (w/a)/(w/a + R/b)$.*

---

[10] This specification is symmetric in labor and capital. Any asymmetry can be absorbed in the distribution of a and b and is immaterial. In extensions below I show that the baseline logic of the model remain unchanged when firms have individual CES production function.

(i) The diffusion in (χ, η) *is*

$$
\begin{aligned}
d\chi_t &= \left[\eta_t(g-\mu_a) - (1-\eta_t)\mu_b + \sigma^2\eta_t(1-\eta_t)\right] dt - \sigma\eta_t\, dW_t^a - \sigma(1-\eta_t)\, dW_t^b, \\
d\eta_t &= \eta_t(1-\eta_t)\left[g + \mu_b - \mu_a + \sigma^2(1-2\eta_t)\right] dt + \sigma\eta_t(1-\eta_t)\left(dW_t^b - dW_t^a\right).
\end{aligned}
$$

(ii) Further, define the transformed variables:

$$
z_t := \log\frac{\eta_t}{1-\eta_t} = \log\frac{w_t}{a_t} - log\frac{R}{b_t},
$$

$$
u_t := \chi_t - \log H(z_t) = \frac{1}{2}\left(\log\frac{w_t}{a_t} + \log\frac{R}{b_t}\right),
$$

with $H(z) := e^{z/2} + e^{-z/2}$. *Then the diffusion in* (u, z) *is*

$$
\begin{aligned}
du_t &= \frac{g-\mu_a-\mu_b}{2}\, dt - \frac{\sigma}{\sqrt{2}}\, dW_{u,t}, \\
dz_t &= (g-\mu_a+\mu_b)\, dt + \sqrt{2}\sigma\, dW_{z,t},
\end{aligned}
\tag{16}
$$

where the Brownian increments are

$$
dW_{u,t} := \frac{dW_t^a + dW_t^b}{\sqrt{2}}, \qquad dW_{z,t} := \frac{dW_t^b - dW_t^a}{\sqrt{2}}.
$$

Since W^a *and* W^b *are independent,*

$$
dW_{u,t}^2 = dW_{z,t}^2 = dt, \qquad dW_{u,t}\, dW_{z,t} = \frac{1}{2}(dW_t^a + dW_t^b)(dW_t^b - dW_t^a) = 0.
$$

Thus, W_u *and* W_z *are independent Brownian motions.*

Proof. Apply Itô's lemma to the functions χ and η. The equation for z follows directly by subtracting $d\log(R/b)$ from $d\log(w/a)$. Equivalently, $d\eta$ follows by applying Itô's formula to $\eta(z) = 1/(1+e^{-z})$ and using $(dz)^2 = 2\sigma^2 dt$. Use that $(dW^a)^2 = (dW^b)^2 = dt$, $dW^a dW^b = 0$ and that all other higher-order terms are zero. □

Lemma 4 demonstrates how firms' cost and labor share evolve. Increments in both log cost, $d\chi$, and labor share, $d\eta$, depend only on η analogous go Gibrat's law. Cost of firms depends negatively on improvements in labor productivity μ_a, in proportion to the labor share η, and analogously for shocks to labor productivity dW^a. Trends and shocks to capital productivity have weight $1-\eta$. The central difference between firms with low and high labor share is that the average growth rate scales with η such that growth in wages pushes cost up for labor-intensive firms (high η) and less so for firms with low labor share. Firms that rely more on the scarce factor, labor, face a continuous cost push rending them continuously less competitive and pushing them towards smaller size and eventual exit.

Second, the labor share is pushed up by trend and shocks to capital productivity and down by labor productivity. With no asymmetry between capital and labor: $g = 0$ and $\mu_a = \mu_b$, the labor share would be pulled towards 1/2 (negative drift for $\eta > 1/2$ and positive for $\eta < 1/2$) but growing wages exert an upward push on the labor share for all firms. The central mechanism of this paper is the balancing between these two effects: Firms face continuous upward pressure on their labor from growing wages, and those firms with high labor share face a continuous cost-disadvantage.

Increments to χ and η are correlated. For a labor intensive firm, $\eta > 1/2$, shocks to labor productivity matter more to cost than those for capital productivity. Since a positive shock to labor productivity pulls down both cost and the labor share, η and χ are positively correlated when $\eta > 1/2$ and negatively correlated when $\eta < 1/2$

The interplay between $d\chi$ and $d\eta$ illustrate the dynamics of the model. Consider a given slice of firms at some χ. If $g + \mu_b - \mu_a > 0$ they are all face a push towards a labor share higher than 1/2. However, the higher is the labor share, the faster firms are pushed up along the χ which affects the stationary distribution along a given slice χ.

Though the central economics are captured by the processes for $(d\chi, d\eta)$, the dependency between the two makes the system difficult to solve. Consequently, I consider a transformed system. First, let $z = log(w_t/a_t) - log(R/b_t)$ be the relative relative labor cost of a firm. It has domain $z \in (-\infty, \infty)$ and takes the value $z = 0$ when $\eta = 1/2$. Second, define $u = \frac{1}{2}\left(log(w_t/a_t) + log(R/b_t)\right)$, as the log average of cost-adjusted productivities. z is therefore related to η and capture differences in labor and capital productivity, whereas u captures the overall level of productivity and relates to the cost. The transformed system in (dz, du) has the convenient property of independence between dz and du (equation 16). The drift of u and z are both pushed up by the cost in labor, $g - \mu_a$, but are asymmetrically affected by growth in capital-augmenting technology, μ_b: Growth in μ_b pushes the cost down but increase the labor share.

At each point in time firms make flow profits as a function of χ, given by $Y_t\Pi_t$ given by (14). I next turn to the question of when firms choose to shut down.

3.3 The exit decision of firms.

Firms exit when the continuation value of operation becomes negative. They discount profits at rate r. On the balanced growth path output grows at the rate g: $Y_t = Y_0e^{gt}$, so normalizing by FY_0 gives the effective discount rate $\rho = r - g$. With normalized flow profits

$F(e^{(1-\nu)(\chi-\chi_0)}-1)$, the normalized incumbent value , v, is

$$v(\chi,\eta):=\sup_{\tau}\mathbb{E}_{\chi,\eta}\left[\int_0^{\tau}e^{-\rho t}\left(e^{(1-\nu)(\chi_t-\chi_0)}-1\right)dt\right],\qquad V(\chi,\eta)=FY_0v(\chi,\eta). \tag{17}$$

The supremum is taken over stopping times with respect to the firm-level productivity filtration. Let $\mathcal{L}_{\chi\eta}v$ denote the generator of the (χ,η) process acting on v, that is how the normalized value depends on (χ,η) and consequently on (a,b). Its exact expression is given in Appendix 6.3. In the continuation region,

$$rv=gv+\mathcal{L}_{\chi\eta}v+e^{(1-\nu)(\chi-\chi_0)}-1,$$

where the left hand side, rv, is the required return on the asset, gv is growth in value from a growing economy, $\mathcal{L}_{\chi\eta}v$ is the movement in v as a consequence of movements in χ and η, and $e^{(1-\nu)(\chi-\chi_0)}-1$ is the (normalized) flow profits.

The exit decision is the obstacle problem

$$\min\left\{\rho v-\mathcal{L}_{\chi\eta}v-\left(e^{(1-\nu)(\chi-\chi_0)}-1\right),\ v\right\}=0. \tag{18}$$

The continuation and stopping regions are

$$\mathfrak{C}:=\{(\chi,\eta):v(\chi,\eta)>0\},\qquad \mathfrak{S}:=\{(\chi,\eta):v(\chi,\eta)=0\}.$$

Unlike Luttmer (2007), It is not possible to give an explicit expression for the value function, but it is possible to characterize the relevant stopping region. I do so in Lemma 5

Lemma 5. *Let v be the value function in (17) and suppose it solves the obstacle problem (18). For each $\eta\in(0,1)$, the map $\chi\mapsto v(\chi,\eta)$ is weakly decreasing. Define the dynamic exit boundary by*

$$\bar{\chi}(\eta):=\sup\{\chi\in\mathbb{R}:\ v(\chi,\eta)>0\}.$$

Then the continuation and stopping regions are represented by the cutoff graph

$$\mathfrak{C}=\{(\chi,\eta):\chi<\bar{\chi}(\eta)\},\qquad \mathfrak{S}=\{(\chi,\eta):\chi\geq\bar{\chi}(\eta)\},$$

up to boundary points. This cutoff graph is constant on a balanced growth path. A firm with state (χ,η) exits when (χ,η) are no longer in this region.

Moreover, $\bar{\chi}(\eta)>\chi_0$ for every $\eta\in(0,1)$. Thus the option value raises the dynamic exit boundary above the static zero-profit point: firms continue even when current normalized profits are negative. Since current profits depend only on χ, while the law of future χ depends *on η, the composition state enters the exit rule only through the boundary $\bar{\chi}(\eta)$.*

Proof. See Appendix 6.6. □

As in Luttmer (2007), firms are willing to bear negative flow profits for the option value of recovering future positive profits and $\bar{\chi}(\eta) > \chi_0$.[11] The exit value is generally not independent of η, but is time invariant. In simulations $\bar{\chi}(\eta)$ is lowest around the center of η: Firms with extremes in labor share η, predominantly rely on productivity shocks to one factor and the variance of their cost shocks is therefore higher. A higher variance implies a larger option value, and they are willing to occur higher negative before shutting down.

Figure 3 illustrates the local law of motion in (χ, η) space in a phase diagram. At each point, the arrow is the deterministic drift of the firm's state: it shows the direction in which cost and the labor share move absent the Brownian shocks, and the size of the arrow shows how quickly firms are moving in this direction. The figure is drawn for $\mu_a = \mu_b = 0$ and $g > 0$. Firms with low η therefore face no deterministic movement in their cost, whereas firms with higher η get pulled towards the cutoff. The ovals represent the stochastic element, a region where "typical" shocks land: for $\eta > 1/2$, η and χ are positively correlated, and the ovals lean upwards, with the reverse for $\eta < 1/2$. Once firms move down to the cutoff region they exit permanently. The figure also shows the χ_0 value which is everywhere lower than $\chi_0(\eta)$. For $\mu_b > 0$ firms with low labor share would be moving to the left as their cost decline.

To summarize: Existing firms move around in (χ, η) space as a consequence of deterministic improvements to technology and individual shocks to technology. At each point in time they make profits as a consequence of their current cost, and if (χ, η) reaches a point where the value function is zero, they irreversibly shut down.

3.3.1 Entry through imitation

Entry occurs through imitation. A potential entrant pays an entry cost, draws an active incumbent as a target. The entrant copies the incumbents technology imperfectly, and both

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[11] To develop some intuition for the $\bar{\chi}(\eta)$ schedule, one can derive a local approximation of $\bar{\chi}(\eta)$ around the static zero profit point to get:

$$\bar{\chi}(\eta) - \chi_0 \approx \frac{\sigma^2}{4b_0} - \frac{\sigma^2 (g + \mu_b - \mu_a)}{4b_0^2}\left(\eta - \frac{1}{2}\right) + \frac{\sigma^2}{4b_0^3}\left(4b_0^2 + \sigma^2 b_0 + (g + \mu_b - \mu_a)^2\right)\left(\eta - \frac{1}{2}\right)^2, \quad b_0 = \frac{g - \mu_a - \mu_b}{2} + \frac{\sigma^2}{4} > 0.$$

The term $\sigma^2/(4b_0) > 0$ demonstrates that firms are willing to bear higher costs than those of zero profits due to the option value. This value is increasing in volatility σ^2. The second term shows that for $g - \mu_a + \mu_b > 0$, firms with higher labor share exit earlier, because firms with higher labor share suffer faster upward push on cost. The quadratic term shows that firms with more centered labor shares ($\eta \approx 1/2$) exit sooner. With a balanced cost structure, firm productivity shocks depend on both (independent) labor– and capital productivity shocks and thus have lower variance in cost shocks.

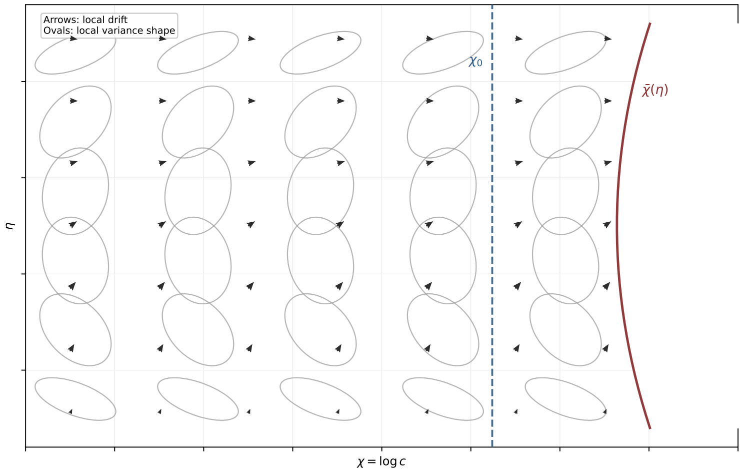

Figure 3: Drift and local covariance in (χ, η) space for $\mu_a = \mu_b = 0$ and $g = 0.02$. Arrows show the local drift of firms, while ellipses show the local covariance geometry of the productivity shocks. The dashed vertical line is the zero profit cutoff, χ_0, and the curve is the $\bar{\chi}(\eta)$ boundary.

labor- and capital-augmenting productivity are shifted down by the same factor $e^{-\Delta}$. Hence the entrant inherits the parent's factor composition and labor share, but enters with a higher unit cost, implying that new entering firms are smaller than existing firms. In (χ, η) space this is a horizontal shift:

$$(\chi, \eta) \quad \longmapsto \quad (\chi + \Delta, \eta), \qquad \Delta > 0,$$

such that entrants have higher costs and revenues that are $(\nu - 1)\Delta$ lower than their parent.[12]

Not all potential entrants survive and it is necessary to distinguish between attempted and successful entry. A parent that is comfortably inside the continuation region can generate a child that also continues. A parent close to the exit boundary may generate a child whose shifted state lies to the right of $\bar{\chi}(\eta)$, in which case the child exits immediately and has zero value. Equivalently, only parents with $(\chi + \Delta, \eta)$:

$$\chi + \Delta < \bar{\chi}(\eta)$$

produce continuing entrants. The imitation wedge therefore has two effects: it lowers the quality of entrants conditional on entry, and it reduces the set of incumbents from which a viable entrant can be copied.

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[12] This specification implies that the labor share of entrants matches that of their parent. Though a natural baseline, a plausible extension would be to allow entrants to direct their search towards firms with a lower labor share and therefore higher continuation value. I do not pursue that here.

In the stationary distribution, this mechanism is the source of new firms. Entry injects mass at states that are shifted to the right of the incumbents they imitate, while exit removes mass at the boundary. The balance between these two margins pins down the cross-sectional distribution of active firms.

3.4 The Stationary Distribution

I now turn from the individual stopping problem to the cross-sectional distribution of active firms. It is simpler to solve the problem in terms of the transformed variables (u, z) where $z_t \equiv \log \frac{\eta_t}{1-\eta_t} = \log \frac{w_t}{a_t} - \log \frac{R}{b_t}$ is a transformation of the labor share and equals the log difference in factor cost. The variable u_t is given by $u_t \equiv \chi_t - \log H(z_t) = \frac{1}{2}\left(\log \frac{w_t}{a_t} + \log \frac{R}{b_t}\right)$, with $H(z) := e^{z/2} + e^{-z/2}$. It captures the level of cost. As stated in Lemma 4, du and dz are independent. The continuation region takes an analogous form:

$$\mathfrak{C} = \{(u, z) : u < \bar{u}(z)\}.$$

Such that the value of u that ensures continuation continues to depend on z. Given independence of movement in u and z this is the only mechanism that introduces dependence between (u, z) though this dependence disappears for $u \to -\infty$ and simulations show it to be small.

Normalized profits are then given by

$$\Pi(u, z) = \kappa_\nu e^{(1-\nu)u} H(z)^{1-\nu} - F. \tag{19}$$

Let $f_{u,z}(u, z)$ be the normalized density of active firms and M is their mass, the price normalization $P = 1$ requires

$$\nu \kappa_\nu M \iint_{\mathfrak{C}} e^{(1-\nu)u} H(z)^{1-\nu} f_{u,z}(u, z)\, du\, dz = 1. \tag{20}$$

In sum, the system in (u, z) behaves like that of (χ, η) but with simpler diffusion processes for du and dz.

3.4.1 The stationary density for a candidate growth rate

The imitation rate for (u, z) copies that for (χ, η) such that a parent produces a parent (u, z) produces a child at $(u + \Delta, z)$. I start out by fixing a candidate growth rate g and a flow rate of *attempted* entry λ_E. Let $m(u, z)$ be the mass density of active firms such that $M := \iint_{\mathfrak{C}} m(u, z)\, du\, dz$, with a density of $f_{u,z}(u, z) := m(u, z)/M$. The attempt rate per incumbent is

$$\varepsilon_A := \frac{\lambda_E}{M}. \tag{21}$$

The stationary density can be solved for using the Kolmogorov forward equation (KFE). Consider the change in density for a given point (u, z) only as a consequence of movements of existing firms. It is given by:

$$\mathscr{L}^* f(u,z) = -\left(\frac{g - \mu_a - \mu_b}{2}\right)\partial_u f - (g + \mu_b - \mu_a)\,\partial_z f + \frac{\sigma^2}{4}\partial_{uu} f + \sigma^2 \partial_{zz} f. \tag{22}$$

The convenience of the system in (u, z) is that the variance terms are constant and that the covariance terms are negative, which simplifies the KFE. To see the intuition for this, consider a small box of size $[u, u+h] \times [z, z+h]$ where h is "small". The mass of firms at this point is approximately $f(u, z)h^2$. All firms face deterministic drift along the u direction of $(g - \mu_a - \mu_b)$. If this is positive firms are moving to the right in the u direction. If $\partial_u f > 0$ more firms are leaving $f(u, z)$ and moving to $f(u + h, z)$ than are leaving $f(u - h, z)$ and move to $f(u, z)$. The difference is captured by the first term of $\mathscr{L}^* f(u, z)$. The quadratic term captures the effect of volatility: firms move around due to random perturbations: If $\partial_u f > 0$ a given point loses more through its left boundary than it gains. If $\partial_{uu} f = 0$ then the net loss to the left equals the net gain from the right and volatility has no effect. When $\partial_{uu} f > 0$ there is net gain from the right. An analogous argument holds for the z direction.

The total change in density has to incorporate the entry of new firms such that stationarity requires:

$$0 = \partial_t f = \mathscr{L}^* f(u,z) + \varepsilon_A f(u - \Delta, z), \qquad (u,z) \in \mathfrak{C}, \tag{23}$$

since firms that enter at (u, z) must have drawn a parent with lower cost from $(u - \Delta, z)$. The first term moves incumbent firms through state space. The second term is the non-local source from imitation. Absorbing exit imposes $f(\bar{u}(z), z) = 0$, with natural decay as $u \to -\infty$ and $|z| \to \infty$, and normalization $\iint_{\mathfrak{C}} f = 1$.

3.4.2 A candidate density in the tail

The movements in (u, z) are independent and dependency on the whole distribution arises only from the curved exit $\bar{u}(z)$. Deep in the tail of $u \to -\infty$ the two will therefore be separable, and the density is governed by the constant-coefficient part of (23). Conjecture that the tail of the size distribution is Pareto which implies that transformed cost follow $e^{\zeta u}$ for some $\zeta > 0$. Separability then gives a candidate solution as

$$f(u,z) \sim e^{\zeta u}\varphi_\zeta(z), \qquad u \to -\infty,$$

for some distribution $\varphi_\zeta(z)$. Substitution this expression in (23) gives

$$\sigma^2\varphi''_\zeta(z) - (g+\mu_b-\mu_a)\varphi'_\zeta(z) + \left[\frac{\sigma^2}{4}\zeta^2 - \frac{g-\mu_a-\mu_b}{2}\zeta + \varepsilon_A e^{-\zeta\Delta}\right]\varphi_\zeta(z) = 0,$$

The imitation source contributes $e^{-\zeta\Delta}$ because an entrant arriving at u is copied from a parent at $u-\Delta$.

To solve for φ, define $\omega_\zeta(z)$ by applying the tilt $\varphi_\zeta(z) = e^{\left((g+\mu_b-\mu_a)/(2\sigma^2)\right)z}\omega_\zeta(z)$: This removes the first-derivative term and leaves

$$\omega''_\zeta(z) - m(\zeta)^2\omega_\zeta(z) = 0, \tag{24}$$

where

$$m(\zeta)^2 = -\frac{\zeta^2}{4} - \frac{\zeta}{2\sigma^2}(\mu_a+\mu_b-g) - \frac{\varepsilon_A}{\sigma^2}e^{-\zeta\Delta} + \frac{(g+\mu_b-\mu_a)^2}{4\sigma^4}, \tag{25}$$

as the characteristic equation. $m(\zeta)^2 = 0$ generally has 0,1 or 2 solutions. In the following I impose the following conjecture / computational lemma

Conjecture 6. *(Computational Lemma) Any compact distribution in (u, z) converges to the unique double-root, ζ^*, of $m^2(\zeta) = 0$*

Proof. Appendix 6.7 gives a conditional proof sketch of this double-root selection. The argument proves the frozen far-left weighted-operator algebra and shows that, conditional on the finite-window residual bounds, endpoint controls, nondegenerate weighted mass, and the critical-entry principle, an interior maximizer of the admissible frontier satisfies the double-root equations. Simulations from many different starting points converge to this point, and the appendix makes explicit which remaining estimates are required for a full theorem. □

I present the double-root (6) as a conjecture: In general the distribution admits solutions with ζ_- and ζ_+ as the double root of (25). In this case the distribution is of the form $f(u,z) \sim \left(C_1e^{\zeta_- u} + C_2e^{\zeta_- u}\right)\varphi_\zeta(z)$ for constants C_1 and C_2. However, the conjecture states that any compact distribution converges to the double-root solution. Heuristically, the double-root condition says that the economy settles on the thinnest tail that can be sustained in a stationary equilibrium. Starting from a compact distribution, there is no pre-existing fat tail to inherit, so the long-run tail is generated by the balance between drift, diffusion, imitation, and exit. The double root is the critical point of that balance: a thinner tail would die out, while thicker tails are not selected by the dynamics. Monte Carlo simulations from different starting points always converge to this point. [13]

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[13] Luttmer (2007) faces an analogous challenge and provides a proof of the double-root selection in a special case of his model.

Under conjecture 6, the relevant double root is given by $m(\zeta)^2 = 0$ together with $\partial_\zeta m(\zeta)^2 = 0$. These two equations give

$$\zeta^* = \frac{g-\mu_a-\mu_b}{\sigma^2} - \frac{1}{\Delta} + \sqrt{\frac{(g-\mu_a-\mu_b)^2}{\sigma^4} + \frac{1}{\Delta^2} + \frac{(g+\mu_b-\mu_a)^2}{\sigma^4}}.$$

When $m(\zeta)^2 = 0$,(24) gives $\omega_\zeta'' = 0$, such that $\omega_\zeta(z) = C_1 + C_2 z$. Since z has infinite domain $C_2 = 0$ and $\omega_\zeta(z) = C_1$ such that:

$$\varphi(z) \propto e^{\left((g+\mu_b-\mu_a)/(2\sigma^2)\right)z}$$

I use this to establish the properties of the stationary distribution in the following proposition

Proposition 7. *The stationary density in $f_{u,z}(u,z)$ for $u \to -\infty$ is separable and given by*

$$f_{u,z}(u,z) \propto e^{\zeta u} e^{\left(\frac{g+\mu_b-\mu_a}{2\sigma^2}\right)z},$$

and in cost $\chi = log(c)$ and labor share $\eta = \frac{1}{1+e^{-z}}$ the stationary density for $\chi \to -\infty$ is separable given by:

$$f_{\chi,\eta}(\chi,\eta) \propto e^{\zeta\chi}\eta^{\alpha-1}(1-\eta)^{\beta-1}, \tag{26}$$

where:

$$\zeta = \frac{g-\mu_a-\mu_b}{\sigma^2} - \frac{1}{\Delta} + \sqrt{\left(\frac{(g-\mu_a-\mu_b)^2}{\sigma^4} + \frac{1}{\Delta^2} + \frac{(g+\mu_b-\mu_a)^2}{\sigma^4}\right)} \tag{27}$$

and

$$\alpha = \frac{\zeta}{2} + \frac{g+\mu_b-\mu_a}{2\sigma^2} \qquad \beta = \frac{\zeta}{2} - \frac{g+\mu_b-\mu_a}{2\sigma^2},$$

Tail revenue is Pareto distributed with exponent $\frac{\zeta}{\nu-1}$ and the labor share out factor cost in the tail is given by:

$$\frac{\alpha}{\alpha+\beta} = \frac{1}{2} + \frac{g+\mu_b-\mu_a}{2\sigma^2\zeta} \tag{28}$$

Proof. Follows from above and a transformation from (u,z) to (χ,η) □

Proposition 7 is the main result of this section. It replicates the properties of the distribution in the static model, but as a result of gradual technological idiosyncratic development for individual firms.The tail parameter of the cost distribution is given by ζ, which balances the influences of growth for individual firms, the growth rate of the economy, the imitation of new firms and the volatility of existing firms. First, ζ is declining in σ^2: more volatility spreads out the distribution in cost and makes the tail fatter. It is increasing in $1/\Delta$:

the easier imitation by new firms is the more compressed the distribution of cost is. Further growth in the economy pulls cost up for existing firms and compresses the distribution whereas underlying drift has the opposite effect.

The labor share distribution is beta, and whether this is bigger or smaller than 1/2 is entirely given by the sign of $g + \mu_b - \mu_a$. Growth in the economy pushes the labor share distribution above 1/2 since all firms face higher labor cost, and the net effect of incumbent negative growth $(\mu_a - \mu_b)$ pushes in the opposite direction. Of course, the growth rate of the economy, g is still an endogenous object, but some intuition can be cleaned from considering the special case where imitation is difficult (Δ is large) such that new firms are considerably smaller than the parents they draw from. Let $q = (g + \mu_b - \mu_a)/(g - \mu_a - \mu_b)$ and take a first order Taylor expansion in $1/\Delta$ around $1/\Delta = 0$ to get:

$$\frac{\alpha}{\alpha+\beta} = \frac{1}{2} + \frac{q}{2\left(1+\sqrt{1+q^2}\right)} + \frac{q}{2(g-\mu_a-\mu_b)\left(1+\sqrt{1+q^2}\right)^2}\frac{\sigma^2}{\Delta} + O(\Delta^{-2}), \tag{29}$$

such that for difficult imitation, the labor share is independent of σ^2. Further, slow incumbent growth in capital productivity ($\mu_b \approx 0$) gives $q \approx 1$ and therefore a labor share for low χ firms of:

$$\frac{\alpha}{\alpha+\beta}|_{\mu_b=0,\Delta\to\infty} = \frac{1}{\sqrt{2}} \approx 0.707,$$

irrespective of growth, g, labor productivity growth μ_a and volatility. The labor share out of revenue of large firms would then be $\frac{\nu-1}{\nu}/\sqrt{2}$.

The reason is that, at this point two effects exactly balance: Consider the labor share distribution for a given (low) χ: Lemma 4 and Figure 3 establish that a higher g push the labor share of all firms with χ up. However, a faster growth rate in the economy and therefore in wages, also pull firms with higher labor share rightwards along the χ dimension faster. These two effects balance and the growth rate of the economy does not affect the labor share. A similar argument (with reverse sign) can be made for a higher μ_a. Higher volatility dampens the push up along the η line, but also dampens the push along the χ line.

Before I introduce the free-entry condition which pins down the growth rate, I briefly relate the current model to the iso-density rays of the static model.

3.4.3 Relation to iso-density rays from the Static model

In the following, I relate the solutions above to the iso-densities of the static model, and give a graphical argument for the $1/\sqrt{2}$ labor share. First, use the tail expression for $f_{\chi,\eta}$ from (26) and use (10) to rewrite the density in $(\tilde{a}, \tilde{b})$ with$\tilde{}$s continuing to represent cost-adjusted productivities ($\tilde{a} = a/w$ and $\tilde{b} = b/R$). This gives

$$g_{\tilde{a},\tilde{b}} \propto \frac{1}{\tilde{a}\tilde{b}} \left(\tilde{a}^{\alpha}\tilde{b}^{\beta}\right)^{-1},$$

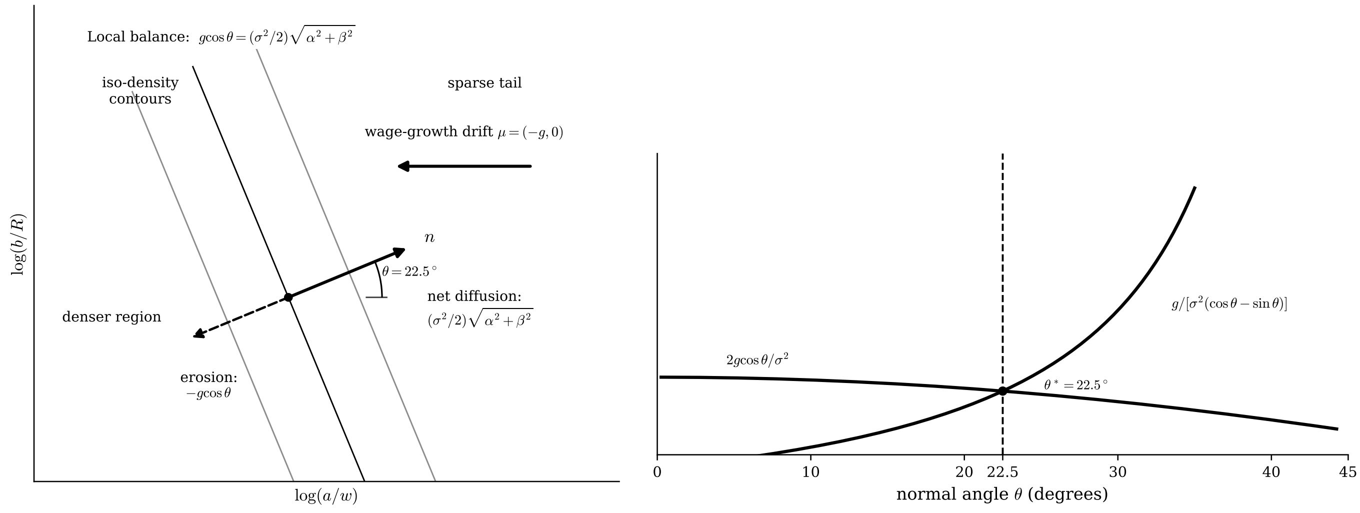
that is the tail of the density is of the form $\psi(x) \propto x^{-1}$ and satisfies Assumption 2. It is independent Pareto. The density is stationary in $(\tilde{a}, \tilde{b})$ but not in (a, b), and the iso-densities satisfy $\alpha log\tilde{a} + \beta log\tilde{b} =$ constant, and at any given point in time therefore also in $(loga, logb)$: $\alpha loga + \beta logb =$ constant$+ \alpha logw - \beta logR$. Figure 4 illustrates the straight iso-density curves in $(log(a/w), log(b/R)$ space. The slope of the iso-densities is given by $-\alpha/\beta$, and the density declines along the normal of these iso-density curves. The (unnormalized) normal vector is $q = (\alpha, \beta)$ which has a length of $\sqrt{\alpha^2 + \beta^2}$, which captures how fast the density declines orthogonal to iso-density curves. The figure also gives the angle of this normal vector, θ. The steeper is this angle the higher is the labor share. The vector q can then be written as $\sqrt{\alpha^2 + \beta^2} \times (\cos\theta, \sin\theta)$, implying $\theta = \arctan(\beta/\alpha)$.

Continue to let Δ be high such that the tail is predominantly determined by internal movements. First, consider the case of $\mu_a = \mu_b = 0$ such that there is symmetric zero drift in technology. Without growth in w the symmetry would imply a slope of $-\alpha/\beta = -1$ and equal labor and capital share. However, even without drift in technology for incumbents, growth in wages implies a negative drift in $\tilde{a}$ and therefore a horizontal leftward push which increases the slope of the curve pushing the labor share up. How strong this effect is, depends on the labor share and therefore the slope of the iso-density curves. If it is steep – the labor share is high – the leftward horizontal push is strong. Projecting the push on the iso-density gives $-g\cos(\theta)$ where θ is the angle of the normal vector as displayed in the picture. The higher is the angle the weaker the push.

The countervailing force is the idiosyncratic shocks to productivity which pushes firms from the denser regions in the southwest out. The countervailing force is proportional to size of shocks and how fast the density declines: $\frac{\sigma^2}{2}\sqrt{\alpha^2 + \beta^2}$. Stability then requires:

$$g\cos(\theta) = \frac{\sigma^2}{2}\sqrt{\alpha^2 + \beta^2} = \frac{1}{2}\frac{g}{\cos(\theta) - \sin(\theta)}.$$

The left hand side is the leftward push from growing wages. It is monotonically decreasing in θ, and therefore the labor share. The right hand side is increasing in θ: When θ is close to 45^{o}, α and β must be very large to ensure $\alpha - \beta = g/\sigma^2$. This implies a unique θ, and solving gives $\theta = 22.5^{o}$, which is illustrated in the figure. Changes in both σ^2 and g affect both forces equally and leave the angle unchanged. This picture is unchanged for $\mu_a \neq 0$ since g moves one for one with changes in μ_a. A $\mu_b > 0$ pushes density up vertically and steepens



(a) (b)

Figure 4: The slope of the iso-density curves in the dynamic model

the curve thereby increasing the labor share as shown in (29). This reduces the angle θ. Further, a finite Δ permits new entrants to affect the tail: This increases the countervailing force, and pushes the curve up in panel B, which gives an intersection at a lower angle and therefore a higher labor share. Thus $\alpha/(\alpha+\beta) = 1/\sqrt{2}$ can therefore be seen as a lower limit of the more general case of $\mu_b > 0$ and $\Delta < \infty$, consistent with equation (29).

3.4.4 Closing the model through free entry

The analysis above treats the growth rate g as given. This is useful because, for a candidate g, the firm problem, the exit boundary, and the stationary distribution can all be characterized before imposing entry. The model is closed by requiring potential entrants to be indifferent between entering and not entering. an entrant pays a fixed entry cost $c^E Y_t$, so that c^E is the stationary entry cost in units of detrended output. Again, the proportionality with output is made for analytical convenience, as this delivers a constant mass of incumbents. It is possible to solve the model for an entry cost not proportional to Y. That would introduce an additional source of growth from expanding variety, but would not alter the overall structure of the model.

For each candidate g, the household block gives $R(g) = \rho + g + \delta$. The the detrended incumbent value is $v_g(u, z)$ and the continuation set $\mathfrak{C}_g$. The Kolmogorov equation gives the stationary density $f_g(u, z)$ of active incumbents. A potential entrant draws an incumbent from this density and enters as a lower-quality copy: a parent at (u, z) produces a child at $(u + \Delta, z)$. If the copied child is outside $\mathfrak{C}_g$, it exits on arrival and has zero value. The value of a potential entrant is therefore

$$\bar{v}_E(g) = \iint_{\mathfrak{C}_g} v_g(u+\Delta, z) f_g(u,z)\, du\, dz. \tag{30}$$

where $v_g(u+\Delta, z)$ is understood to be zero when $(u+\Delta, z) \notin \mathfrak{C}_g$. Equivalently, using the shift notation defined above,

$$\bar{v}_E(g) = \iint_{\mathfrak{C}_g} (T_\Delta v_g)(u,z) f_g(u,z)\, du\, dz.$$

Free entry closes the model:

$$c^E = \bar{v}_E(g). \tag{31}$$

The economic force behind this condition is simple. A higher growth rate raises wages relative to incumbent productivities. This pushes firms toward higher cost states, lowers the continuation value of operating firms, and makes the copied entrant less valuable on average. At the same time, entry is costly in units of output. The equilibrium growth rate is the value of g at which the expected value of the copied entrant exactly covers the entry cost.

Proposition 8. *Under the tail conditions stated in Appendix 6.5, there exists $\bar{c}_T^E < \infty$ such that, for every $c^E > \bar{c}_T^E$, the free-entry equation $c^E = \bar{v}_E(g)$ has a unique equilibrium solution. This solution lies in the high-entry-cost tail region. On this range, the equilibrium growth rate is decreasing in c^E.*

Proof. See Appendix 6.5. □

Though it is not possible to establish generally that the free-entry condition has a unique g for c^E high enough g is unique. In simulations $\bar{v}_E(g)$ is always decreasing and thus there is a unique g.

Once g^* is selected, the remaining aggregate objects follow from the aggregate closure in the next subsection. The Euler equation gives $r^* = \rho + g^*$ and $R^* = \rho + g^* + \delta$, while the price-index and factor-market equations determine firm mass, output relative to wages, and the capital-output ratio. Thus the free-entry condition supplies the missing scalar equation that turns the candidate-g stationary distribution into a balanced-growth equilibrium.

3.4.5 Aggregate closure and balanced growth

For any density f on $\mathfrak{C}$ define the moment family

$$\mathcal{M}_{q,r}[f] := \iint_{\mathfrak{C}} \eta(z)^q (1-\eta(z))^r \underbrace{e^{(1-\nu)u} H(z)^{1-\nu}}_{e^{(1-\nu)\chi} = c^{1-\nu}} f(u,z)\, du\, dz, \tag{32}$$

such that $\mathcal{M}_{q,r}[f]$ captures the relevant macroeconomic objects like aggregate price level and factor shares. In particular, the price-index condition (20) is

$$1 = \nu\kappa_\nu M \mathcal{M}_{0,0}[f], \qquad M = \frac{1}{\nu\kappa_\nu \mathcal{M}_{0,0}[f]}. \tag{33}$$

Using factor-demand from individual firms gives labor and capital clearing give

$$\bar{L} = (\nu-1)\kappa_\nu \frac{Y}{w} M \mathcal{M}_{1,0}[f], \qquad K = (\nu-1)\kappa_\nu \frac{Y}{R} M \mathcal{M}_{0,1}[f]. \tag{34}$$

Since $\eta(z) + (1-\eta(z)) = 1$, $\mathcal{M}_{1,0}[f] + \mathcal{M}_{0,1}[f] = \mathcal{M}_{0,0}[f]$. Hence

$$\frac{w\bar{L} + RK}{Y} = \frac{\nu-1}{\nu} \cdot \frac{\mathcal{M}_{1,0}[f] + \mathcal{M}_{0,1}[f]}{\mathcal{M}_{0,0}[f]} = \frac{\nu-1}{\nu},$$

the usual markup identity.

Proposition 9. *Suppose $f(u,z;g)$ is stationary in detrended coordinates, labor is fixed at $\bar{L}$, and the rental rate R is constant along the balanced growth path. Then Y/w and K/Y are constant, and w, K, and Y grow at the common rate g.*

Proof. From (34) and (33),

$$\frac{Y}{w} = \frac{\nu\bar{L}\,\mathcal{M}_{0,0}[f]}{(\nu-1)\mathcal{M}_{1,0}[f]}, \qquad \frac{K}{Y} = \frac{\nu-1}{\nu R} \cdot \frac{\mathcal{M}_{0,1}[f]}{\mathcal{M}_{0,0}[f]}.$$

Both ratios depend only on stationary moments of f and on the constant rental rate R. Hence they are constant along the balanced growth path, and K and w both grow with output. □

In sum, there exists a balanced growth path with constant factor shares and Pareto distributed revenue. Further, for large firms the labor share distribution is Beta distributed with parameters pinned down by technological drift and growth rate of the economy. I next turn to the determinants of the growth rate before providing a simulation of the model.

3.5 Determining the growth rate of the economy

I close of the theoretical treatment by examining the determinants of the only endogenous variable in tail shape of (χ, η), the growth rate g. I do so in two steps: First, for general entry cost, and then for the special case of high entry cost, c^E, which fixes the tail index of the χ distribution, ζ at $\zeta/(\nu-1)$ just above 1.

3.5.1 The growth rate of the economy for general fixed cost of entry

First, for both the dynamics of firms, the value function and other relevant objects, the growth rate only enters as $g - \mu_a$, which is uniquely pinned down. Consequently, the growth rate of the economy moves one for one with μ_a. The labor share and the tail index are therefore independent of μ_a for all parameter values. For a given growth rate, a higher μ_a increases the value of entry, and entry must go up to ensure a value function that equates to the cost of entry, which pushes up the growth rate.

The formal statement and proof of this relative-growth invariance are collected in Corollary 1 in Appendix 6.4. It is difficult establishing general comparative statics with respect to other parameters, and instead I focus on the particular case of a Pareto tail of around 1 in revenue as well as simulations.

3.5.2 The growth rate for $c^E \to \infty$

Consider a the case of entry cost being much larger than fixed cost of operation. In that case, the expected value of entry must be large. Revenue and therefore profits of entry are determined by the tail index of the size distribution in revenue and profits given by $\zeta/(\nu-1)$, where value of entry goes to infinity for $\zeta \to (\nu - 1)$, such that a high c^E will have a Pareto tail at ζ^* just above $\nu - 1$. Even outside of this regime the value function is very sensitive to changes in ζ, such that ζ varies very little with changes in parameters, and consequently has little impact on values.

In the regime of high fixed, we must therefore have $\zeta = \bar{\zeta}$ slightly above $(\nu - 1)$. With $\zeta = \bar{\zeta}$ fixed, I solve for g as a function of exogenous parameters (including $\bar{\zeta}$) using (27) to get:

$$g - \mu_a = -\mu_b - \sigma^2\left(\bar{\zeta} + \frac{1}{\Delta}\right) + \sqrt{2\left(\mu_b + \sigma^2\left(\bar{\zeta} + \frac{1}{\Delta}\right)\right)^2 - 2\mu_b^2 - \frac{\sigma^4}{\Delta^2}}, \tag{35}$$

which continues to have $\partial g/\partial \mu_a = 1$.

One can think of g as being the source of three sources of growth. First, consider the special case $\mu_b = 0$ and $\Delta \to \infty$ which gives:[14]

$$g = \mu_a + \left(\sqrt{2} - 1\right)\sigma^2\bar{\zeta}.$$

From this it follows that even for $\mu_a = 0$ and small new firms, there is positive growth in the economy. This comes from the continuous churn of low-productivity firms dying and new

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[14] Luttmer (2007) takes an analogous approach in his model, though his expression differs since uncorrelated shocks affect both labor - and capital productivity.

firms being born, even when new firms are small relative to the tail. With this as a baseline, and taking a Taylor-expansion, I establish:

Proposition 10. *Consider the case of a high fixed cost of entry and a corresponding Pareto tail in revenue,* $\bar{\zeta}/(\nu-1)$ *close to, but above* 1*. The growth rate of the economy given by (35) approximately equals:*

$$g \approx \underbrace{\mu_a}_{\textit{Labor-saving technology}} + \left(\sqrt{2}-1\right)\left\{\underbrace{\mu_b}_{\textit{Capital-saving technology}} + \underbrace{\sigma^2\bar{\zeta} + \frac{\sigma^2}{\Delta}}_{\textit{Entry}}\right\},$$

where growth increases one-for-one with labor-augmenting technology and less than one-for-one with capital augmenting technology. Contribution to growth from entry is positive even when $\Delta \to \infty$ *and is increasing in shocks to growth.*

The labor share of the largest firms approximately equals:

$$\frac{\alpha}{\alpha+\beta} \approx \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}\sigma^2\bar{\zeta}}\mu_b + \frac{(\sqrt{2}-1)}{2\bar{\zeta}}\frac{1}{\Delta}, \tag{36}$$

which takes a minimum in $1/\sqrt{2}$ *for* $\mu_b \geq 0$ *and is increasing in both capital-augmenting technological growth and the relative size of new firms,* $1/\Delta$*.*

Proof. Take a Taylor-expansion of (35) in $(\mu_b, 1/\Delta)$ around $(\mu_b, 1/\Delta) = (0, 0)$. Use this in the expression for $\alpha/(\alpha+\beta)$ in (28) to get the expression for $\alpha/(\alpha+\beta)$ □

This gives the three sources of growth in the economy. Labor-saving technology increases the growth rate one-for-one, whereas capital-saving technology has a less than one-for-one ($\sqrt{2}-1 \approx 0.41$) effect on productivity. The reason is that whereas increases in a makes the scarce factor more productivity, increases the productivity of the abundant factor. Finally, entry increases the overall growth rate and this effect is higher if Δ is low such that new firms are larger.

The labor takes a minimum value in $1/\sqrt{2}$ and is increasing in μ_b and $1/\Delta$ (to first order). Consider an increase in μ_b which increases growth but not 1-for-1. The reason is that labor-augmenting growth benefits the scarce factor and is consequently more important for growth. Thus, the direct effect of a higher μ_b is to push up the labor share, and the indirect effect of pulling high-labor share firms towards the exit is muted. Consequently, the labor share rises. Higher imitation (lower Δ) pushes up the growth rate and therefore the upward drift in the labor share. But since it also replenishes firms along a given χ slice faster the net effect is a higher labor share.

Hence, for relatively low technological improvement of capital and a relatively larger imitation penalty, the model predicts a labor share for the largest firms of:

$$\frac{\nu - 1}{\nu} \frac{1}{\sqrt{2}},$$

which for a markup of 10% delivers a labor share of $0.9/\sqrt{2} \approx 0.64$.

3.6 Extensions

Before turning to the simulations, I briefly discuss four extensions to the model. The first two – a more general shock structure, and CES for individual firms – keep the linear iso-density curves and the associcated Cobb-Douglas production function.[15] For the fourth extension, I provide a sketch of the type of productivity processes required to deliver constant factor shares on the balanced growth path, but an aggregate CES production function for a given technology distribution as in the extension of the baseline model in section 2.5.3.

3.6.1 Correlated and unequal productivity shocks

Suppose more generally that

$$d \log a_t = \mu_a dt + \sigma_a dW_t^a, \qquad d \log b_t = \mu_b dt + \sigma_b dW_t^b, \qquad d\langle W^a, W^b \rangle_t = \rho dt,$$

such that ρ captures the correlation of the two shocks and shocks to a and b are allowed to have different variances. The drift terms in (u, z) are unchanged, but the instantaneous variance in u is now $\left(\sigma_a^2 + \sigma_b^2 + 2\rho\sigma_a\sigma_b\right)/4$, for z it is $\sigma_a^2 + \sigma_b^2 - 2\rho\sigma_a\sigma_b$ and the correlation is $(\sigma_a^2 - \sigma_b^2)/2$. These naturally collapse to the terms in Lemma 4 when $\sigma_a = \sigma_b$ and $\rho = 0$. Appendix 6.8 shows that the tail nevertheless retains the same functional form and the interior tail continues to have linear iso densities. The growth rate and the labor share from Proposition 10 change to:

$$g \approx \mu_a + (\Gamma^{-1} - 1)\mu_b + \sigma_b^2 \left(\Gamma^{-1} + \rho \frac{\sigma_a}{\sigma_b} - 1 \right) \left(\bar{\zeta} + \frac{1}{\Delta} \right),$$

$$\frac{\alpha}{\alpha + \beta} \approx \Gamma + \Gamma \frac{\mu_b}{\sigma_b^2 \bar{\zeta}} + \Gamma^2 \frac{\sigma_b^2 \left(\Gamma^{-1} + \rho \frac{\sigma_a}{\sigma_b} - 1 \right)}{\sigma_b^2 \bar{\zeta}} \frac{1}{\Delta},$$

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[15] A third extension is state-independent exogenous death. If active firms die at Poisson rate λ, the stationary forward equation adds a common killing term $-\lambda f$. This thins the low-cost tail directly, since firms can disappear before they have survived long enough to diffuse far from the exit boundary. The functional form of the distribution survives, but the labor share is lower.

where $\Gamma \equiv 1/\sqrt{\frac{\sigma_a^2}{\sigma_b^2} + 1 - 2\rho\frac{\sigma_a}{\sigma_b}}$ with $\Gamma = 1/\sqrt{2}$ when shocks are uncorrelated and of the same size. When Δ is large, the labor share increases in ρ and declines in σ_a/σ_b.

3.6.2 CES production for individual firms

Suppose individual firms instead have CES production with an elasticity of substitution $\epsilon < 1$. The associated unit cost can be written, up to normalization, as

$$c_\epsilon\left(\frac{w}{a}, \frac{R}{b}\right) = \left[\left(\frac{w}{a}\right)^{1-\epsilon} + \left(\frac{R}{b}\right)^{1-\epsilon}\right]^{1/(1-\epsilon)}.$$

This changes the map from the relative effective cost state to the firm's labor share, but not the law of motion for the underlying states (u, z). In particular, the firm labor share is

$$\eta_\epsilon(z) = \frac{1}{1 + \exp\{-(1-\epsilon)z\}},$$

so a higher firm-level elasticity smooths the response of the labor share to a given change in z.

The key point is that the double-root calculation behind Proposition 10 is unchanged. The CES specification bends iso-cost curves and the exit boundary, but it does not bend the tail iso-density curves of the productivity distribution. Thus, holding $\bar{\zeta}$ fixed, the growth rate remains

$$g \approx \mu_a + \left(\sqrt{2} - 1\right)\left\{\mu_b + \sigma^2\bar{\zeta} + \frac{\sigma^2}{\Delta}\right\}.$$

The mean tail labor share also remains

$$\frac{\alpha}{\alpha+\beta} \approx \frac{1}{\sqrt{2}} + \frac{\mu_b}{\sqrt{2}\sigma^2\bar{\zeta}} + \frac{\sqrt{2}-1}{2\bar{\zeta}}\frac{1}{\Delta}.$$

Therefore firm-level CES substitution does not change the growth-rate or average tail labor-share approximation in Proposition 10. Appendix 6.9 gives the derivation by changing variables from relative effective costs to firm labor shares.

3.6.3 An aggregate elasticity of substitution of 1/2

The dynamic model demonstrates the link between geometric Brownian motions, linear iso-density curves, and the aggregate Cobb-Douglas production function. The extension in 2.5.3 to the static model, shows that a particular hump-shaped distribution along linear curves in $(log(a), log(b))$ delivers an aggregate elasticity of substitution of 1/2. This suggests that mean reversion in $log(a/b)$ can deliver an aggregate elasticity of substitution of less than 1. In the following I give a proof-of-concept showing that this conjecture can be made internally consistent in the interior stationary equation. Fully developing the link between diffusion processes and the aggregate elasticity of substitution is beyond the scope of this paper, and this section should be read as a proof-of-concept.

Consider the baseline case with no constant incumbent technological drift, $\mu_a = \mu_b = 0$, with a constant growth rate in g and a constant R. Add one state-dependent directed-technical-change term and replace the diffusion terms in (u, z) with:

The baseline diffusion in (u, z) is modified only by adding a state-dependent drift in the relative-productivity state. Specifically, replace the diffusion for u and z from Lemma 4 with[16]:

$$\begin{aligned} du_t &= \frac{g}{2}\,dt - \frac{\sigma}{\sqrt{2}}\,dW_t^u, \\ dz_t &= [g + b(z)]\,dt + \sqrt{2}\sigma\,dW_t^z, \end{aligned} \tag{37}$$

where $\hat{z} \equiv z - \bar{z}$ and

$$b(z) = -\frac{\sigma^2(m+1)}{2}\tanh\left(\frac{\hat{z}}{2}\right) + g\left[\cosh^{m+1}\left(\frac{\hat{z}}{2}\right) - 1\right] + \mathcal{A}_m \cosh^{m+1}\left(\frac{\hat{z}}{2}\right)\int_0^{\hat{z}} \cosh^{-(m+1)}\left(\frac{s}{2}\right) ds,$$

where $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, $\cosh^{m+1}(x) = \left(\frac{e^x + e^{-x}}{2}\right)^{m+1}$ and $\mathcal{A}_m \equiv \frac{\sigma^2 m^2}{4} - \frac{gm}{2} + \varepsilon_A e^{-m\Delta} < \sigma^2(m+1)/4$, where m is a constant and should be thought of as $\alpha + \beta$. $\bar{z}$ is a constant and $\hat{z}$ measures deviations from it. The drift satisfies $b(\bar{z}) = 0$, while the condition below gives local mean reversion around $\bar{z}$. The baseline model has $b(z) = 0$ and (37) therefore adds a local mean-reverting force around $\bar{z}$. In addition to the drift to z from growing wages, firms are pulled towards $\bar{z}$.

The additional drift $b(z)$ can be interpreted as reduced-form directed technical change. Since high z means a high labor-cost share, a negative value of $b(z)$ raises labor productivity relative to capital productivity and pushes z back down. Conversely, when z is low, a positive value of $b(z)$ tilts productivity growth toward capital and pushes z upward. Locally around $\bar{z}$,

$$b(z) = \left(\mathcal{A}_m - \frac{\sigma^2\theta}{4}\right)(z - \bar{z}) + o(z - \bar{z}), \tag{38}$$

so the added drift is locally mean-reverting since $\mathcal{A}_m < \sigma^2\theta/4$ by assumption.

Proposition 11. *Suppose $\mu_a = \mu_b = 0$, the rental rate R is constant, wages grow at rate g, and the CES weights satisfy $\lambda_{L,t}^2 w_t = \Lambda_L$ and $\lambda_K^2 R = \Lambda_K$, where Λ_L, Λ_K are constants*

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[16] One can implement this directly in the diffusion processes for (a, b) by letting the drift terms depends explicitly on z: $\mu_a(z) = -\frac{1}{2}b(z)$ and $\mu_b(z) = \frac{1}{2}b(z)$, which directly leads to (37).

and $\bar{z} = log(\Lambda_L/\Lambda_K)$. Let $b(z)$ be the drift defined above and consider the diffusion in (37). Then, away from the absorbing exit boundary, the candidate stationary interior density over productivities has the form

$$\psi_t(a, b) \propto \frac{a^{-1/2}b^{-1/2}}{\left(\lambda_{L,t}^2 a + \lambda_K^2 b\right)^{m+1}}.$$

Consequently, the static interior aggregation result applied to this density gives a CES substitution margin with elasticity of substitution equal to $1/2$*. Along this interior balanced-growth construction, aggregate relative factor payments are constant and satisfy*

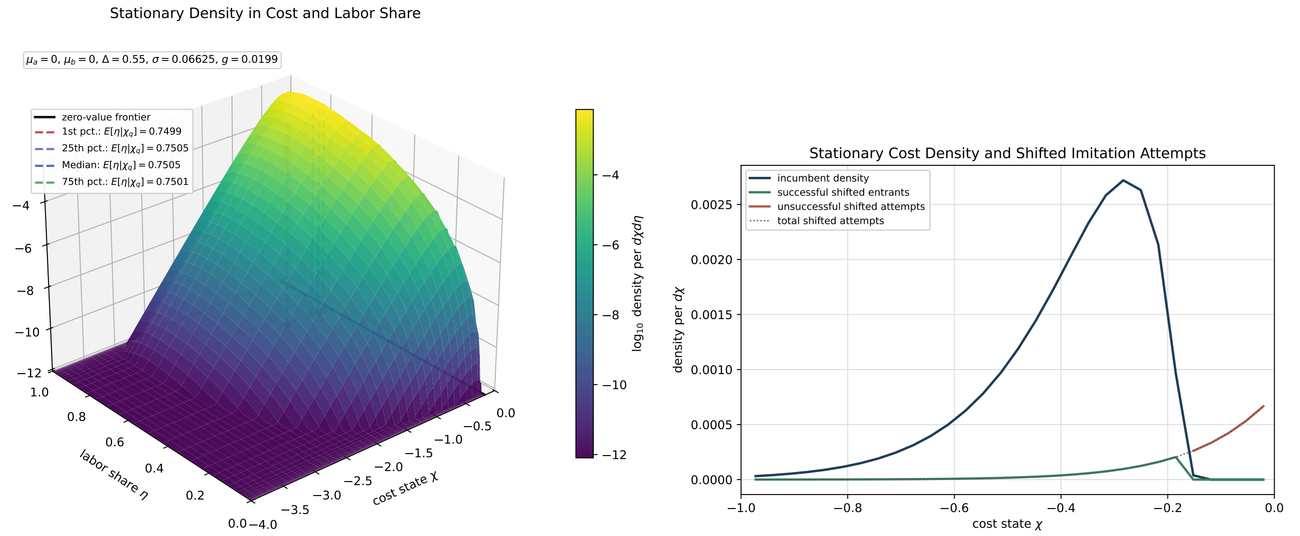
$$\frac{w_t L_t}{RK_t} = \left(\frac{\Lambda_L}{\Lambda_K}\right)^{1/2}$$

The proof is given in Appendix 6.10.

That is, the particular class of diffusions in (37) make the target CES-1/2 density stationary for the interior KFE. Since technological progress is labor-augmenting in the construction, the relative factor-payment ratio is constant along the interior balanced-growth path at $(\Lambda_L/\Lambda_K)^{1/2}$. This is an interior construction: it does not by itself characterize the boundary correction near exit or prove that the global stationary distribution is exactly CES-1/2. The intuition for why can most easily be seen by comparing panel a and panel c in Figure 1. Panel a is the linear iso-density curves which generates a Cobb-Douglas aggregate production function. The mean-reverting diffusion process pushes mass towards $\bar{z}$ and thus creates a hump-shaped density along curves in $(log(a), log(b))$. This breaks the replication argument of the static model, and thus introduces an elasticity of substitution of less than 1. The particular form of the mean-reversion in (37) generates this interior elasticity of 1/2. It would be a valuable path for future research to endogenize the mean-reversion as the consequence of an explicit directed-technical change problem of individual firms, thereby endogenizing the value of $\bar{z}$, which here is taken to be exogenous.

4 Illustrative Simulations

The preceding results characterize the tail analytically. I now use the full numerical solution to check how much of this structure survives away from the asymptotic region. The simulations solve the HJB and stationary KFE on the continuation region, compute the joint distribution of cost and factor composition, and compare the resulting objects with the tail formulas. The exercise is illustrative rather than a full calibration: the focal specification sets $\mu_a = 0$, centers μ_b at zero, and uses $\Delta = 0.55$ with a volatility chosen to generate



(a) The stationary distribution in (χ, η)

(b) The stationary distribution in χ of incumbents and entrants.

Figure 5: Stationary density over cost and factor composition. The surface plots the full stationary density in (χ, η), with the vertical axis equal to the log density per $d\chi d\eta$. The black line is the zero-value frontier. The legend reports the local conditional mean labor share at the first, twenty-fifth, fiftieth, and seventy-fifth percentiles of the stationary χ distribution.

balanced-growth rates around two percent.

Figure 5.a shows the full distribution in (χ, η). Here Δ is finite which is why the labor share is slightly above $1/\sqrt{2}$ consistent with equation (36). The labor share is basically unchanged throughout the distribution. Panel b shows the stationary distribution in χ as well as the distribution of entrants - green for those who enter and red for those who do not. During a small slice of the distribution only a fraction of firms enter, consistent with $\bar{\chi}(\eta)$ being dependent on the η draw. The χ distribution is exponential, and the size distribution therefore Pareto.

Second I show how growth and the tail labor share vary with μ_b around $\mu_b = 0$ in Figure 6. The approximation value is slightly lower than the approximation due to finite Δ, but the match between the approximation and the simulation in panel b for the labor share are very close.

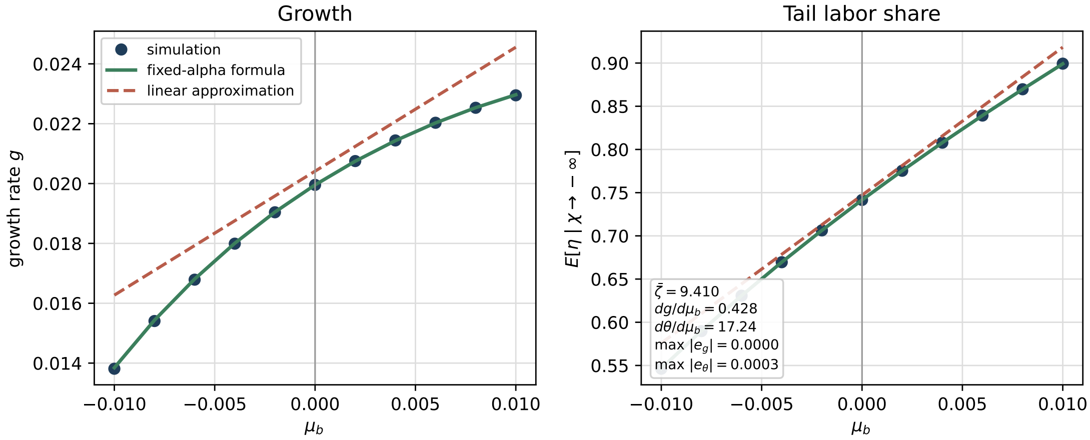


Figure 6: Numerical and approximate responses to μ_b. The figure compares simulated equilibrium growth and the simulated tail labor share with the fixed-$\bar{\zeta}$ formula and the first-order approximation around $\mu_b = 0$. The approximation captures the sign and local magnitude of both responses.

5 Conclusion

This paper shows how an aggregate labor share can be derived from the cross-sectional distribution of firms rather than imposed through an aggregate Cobb-Douglas production function. Firms differ in labor- and capital-augmenting productivity, and selection and entry determine which productivity pairs remain active. The relevant restriction is geometric: when the active density is balanced along rays $\alpha \log a + \beta \log b$, the economy aggregates as Cobb-Douglas and labor's share of factor costs is $\alpha/(\alpha+\beta)$. In the power-law benchmark this same restriction also has firm-level content, generating Pareto size tails, beta-distributed labor shares, and beta-logistic capital-labor ratios.

Firms move in two-dimensional productivity space, selection removes low-value firms, and entrants imitate incumbents imperfectly; on a balanced growth path, exit, entry, and drift generate the linear iso-density structure needed for aggregation. The asymmetry is not a primitive exponent. Capital accumulates while labor is supplied exogenously, so rising wages shift the active distribution toward higher labor shares. With little capital-saving progress, the model approaches the square-root benchmark: a factor-cost labor share near $1/\sqrt{2}$, or about 64 percent of GDP under a 10 percent markup.

The paper provides the Cobb-Douglas aggregate function as a baseline, and provides an illustration of how other productivity dynamics can lead to an aggregate production function with a lower elasticity of substitution. Future research should provide a clearer link between the dynamics of individual firms and the aggregate elasticity of substitution.

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6 Appendix

6.1 Uniqueness of the price level

This appendix proves Proposition 1. The argument reduces the equilibrium to a single equation in the composite variable $z = w^\alpha R^\beta P^{-(\alpha+\beta)}$, then exhibits strict monotonicity of that equation through a homogeneity rescaling. Notably, no monotonicity assumption on ψ is required.

Proof. Define $z = w^\alpha R^\beta P^{-(\alpha+\beta)}$ and $\xi = z^{1/(\alpha+\beta)} = (w^\alpha R^\beta)^{1/(\alpha+\beta)}/P$. Substituting $a = (w/P)\tilde{a}$ and $b = (R/P)\tilde{b}$ in the price-index equation 9, and using homogeneity of degree one of c together with $g(a,b) = (ab)^{-1}\psi(a^{\alpha}b^{\beta})$, the equation reduces to

$$J(z) := \int_{\tilde{a}_0}^{\infty}\int_{\tilde{b}_0(\tilde{a})}^{\infty} \frac{c(1/\tilde{a}, 1/\tilde{b})^{1-\nu}}{\tilde{a}\,\tilde{b}}\,\psi\big(z\,\tilde{a}^{\alpha}\tilde{b}^{\beta}\big)\,d\tilde{b}\,d\tilde{a} = K_{\nu}, \qquad K_{\nu} := \Big(\tfrac{\nu}{\nu-1}\Big)^{\nu-1}.$$

The cutoffs $\tilde{a}_0$ and $\tilde{b}_0(\tilde{a})$ are independent of (w, R, P), so uniqueness of P is equivalent to uniqueness of the z solving $J(z) = K_{\nu}$. Now apply the further change of variables $\tilde{a}' = \xi\tilde{a}$, $\tilde{b}' = \xi\tilde{b}$. Three identities follow: (i) homogeneity of c gives $c(1/\tilde{a}, 1/\tilde{b}) = \xi\, c(1/\tilde{a}', 1/\tilde{b}')$; (ii) the argument of ψ simplifies to $z\,\tilde{a}^{\alpha}\tilde{b}^{\beta} = \tilde{a}'^{\alpha}\tilde{b}'^{\beta}$, eliminating z from inside ψ; (iii) the measure satisfies $d\tilde{a}\,d\tilde{b}/(\tilde{a}\,\tilde{b}) = d\tilde{a}'\,d\tilde{b}'/(\tilde{a}'\,\tilde{b}')$. The active region becomes $\Omega^{*}_{\xi} = \{(\tilde{a}', \tilde{b}') : c(1/\tilde{a}', 1/\tilde{b}') \le \bar{c}/\xi\}$ with $\bar{c} := (F/\kappa_{\nu})^{1/(1-\nu)}$, so

$$J(\xi) = \xi^{1-\nu}M(\xi), \qquad M(\xi) := \int_{\Omega^{*}_{\xi}} \frac{c(1/\tilde{a}', 1/\tilde{b}')^{1-\nu}}{\tilde{a}'\,\tilde{b}'}\,\psi\big(\tilde{a}'^{\alpha}\tilde{b}'^{\beta}\big)\,d\tilde{a}'\,d\tilde{b}'.$$

The integrand of $M(\xi)$ is nonnegative and does not depend on ξ, while the domain Ω^{*}_{ξ} shrinks monotonically as ξ rises; hence $M(\xi)$ is non-increasing. Since $\nu > 1$, the prefactor $\xi^{1-\nu}$ is strictly decreasing. Therefore $J(\xi)$ is strictly decreasing on $(0, \infty)$, with no shape assumption on ψ required. Boundary behavior: as $\xi \downarrow 0$, the active region exhausts the support of ψ and $M(\xi)$ converges to a finite limit by the integrability tail condition, while $\xi^{1-\nu} \to \infty$, so $J(\xi) \to \infty$. As $\xi \uparrow \infty$, Assumption 1 ensures that c is bounded below away from zero, so $\bar{c}/\xi$ eventually drops below this lower bound and $\Omega^{*}_{\xi} = \emptyset$; hence $J(\xi) \to 0$. By strict monotonicity and the intermediate value theorem, there exists a unique $\xi^{*} \in (0, \infty)$ with $J(\xi^{*}) = K_{\nu}$. The corresponding price level is $P = (w^{\alpha}R^{\beta})^{1/(\alpha+\beta)}/\xi^{*}$, so $\kappa_P = 1/\xi^{*}$ is uniquely pinned down. □

6.2 Finite support and the aggregate elasticity correction

This appendix records the local correction when the interior-cutoff assumption fails because a lower productivity support truncates the active set. The calculation is only meant to describe the one-sided margin: it shows how support clipping changes the aggregate substitution margin relative to the unit-elastic interior benchmark.

Let η_i denote firm i's variable-cost labor share and let $\bar{\eta} \equiv wL/(wL + RK)$ denote the aggregate labor share in variable factor payments. Consider the one-sided case in which the relevant support restriction is $a_i \ge a_{\min}$. Define the induced cutoff in labor-share space by

$$\lambda \equiv \eta\bigg(\frac{w/a_{\min}}{R/b_0(a_{\min}; w, R, P)}\bigg),$$

where $b_0(a_{\min}; w, R, P)$ is defined by

$$c\left(\frac{w}{a_{\min}}, \frac{R}{b_0(a_{\min}; w, R, P)}\right) = P(F/\kappa_\nu)^{1/(1-\nu)}.$$

Thus λ is the labor share of the zero-profit ray that intersects the support boundary $a = a_{\min}$. When the $a_{\min}$ bound is the relevant one-sided restriction, zero-profit rays with $\eta_i > \lambda$ are the high-labor-share rays truncated by the support bound.

Let Q_λ denote the revenue-weighted distribution of zero-profit cost-share rays after integrating out radial productivity. Let $\mathcal{M}_\lambda^\partial$ be the set of rays that become constrained under the relevant one-sided marginal change in w/R, and define

$$\pi_m \equiv Q_\lambda(\mathcal{M}_\lambda^\partial), \qquad \eta_m \equiv E_{Q_\lambda}[\eta_i \mid \mathcal{M}_\lambda^\partial].$$

In an already-clipped one-sided branch, $\mathcal{M}_\lambda^\partial = \{\eta_i > \lambda\}$, so $\pi_m = Q_\lambda(\eta_i > \lambda)$. At a knife-edge endpoint, π_m should instead be read as the one-sided exposed revenue weight of the margin that becomes constrained after the price change, not as current missing mass. In the interior branch, $\pi_m = 0$.

Let ζ_Y denote the Pareto coefficient of the firm revenue distribution among active firms:

$$\Pr(Y_i > y \mid i \text{ active}) \sim C_Y y^{-\zeta_Y}.$$

Then the revenue-weighted radial exponent is $(\nu - 1)(\zeta_Y - 1)$. The one-sided deviation from the unit-elastic benchmark is

$$1 - \sigma_{KL}^\partial = \frac{d \log \lambda}{d \log(w/R)} \frac{(\nu - 1)(\zeta_Y - 1)\pi_m(\eta_m - \bar{\eta})}{\bar{\eta}(1 - \bar{\eta})}.$$

For the independent Pareto one-sided branch, where $\zeta_Y = (\alpha + \beta)/(\nu - 1)$, this becomes

$$1 - \sigma_{KL}^\partial = \frac{\beta(\zeta_Y - 1)\pi_m(\eta_m - \bar{\eta})}{[\zeta_Y - (\zeta_Y - 1)\pi_m]\bar{\eta}(1 - \bar{\eta})}.$$

The second expression makes the size of the correction transparent. Holding the exposed revenue weight and factor shares fixed away from their endpoints, the deviation satisfies $1 - \sigma_{KL}^\partial = O(\zeta_Y - 1)$ as $\zeta_Y \downarrow 1$. Hence the support-bound correction is small when the revenue tail is close to Zipf, unless a large exposed margin is clipped. The correction is positive exactly when the one-sided exposed margin has positive revenue weight and is more labor-intensive than the aggregate, $\eta_m > \bar{\eta}$. The case in which the $b_{\min}$ bound truncates low-labor-share rays is symmetric.

6.3 Generator for the incumbent value problem

This appendix gives the exact backward generator used in the incumbent value problem. The state variables are the same as in Lemma 4: $\chi = \log c$ and $\eta = (w/a)/(w/a + R/b)$. Define

$$b_\chi(\eta) \equiv \eta(g-\mu_a) - (1-\eta)\mu_b + \sigma^2\eta(1-\eta), \qquad b_\eta(\eta) \equiv \eta(1-\eta)\left[g+\mu_b-\mu_a+\sigma^2(1-2\eta)\right].$$

For any twice continuously differentiable function $v(\chi,\eta)$, the backward generator is

$$\begin{aligned}\mathcal{L}_{\chi\eta}v(\chi,\eta) &= b_\chi(\eta)v_\chi + b_\eta(\eta)v_\eta \\ &\quad + \frac{\sigma^2}{2}\left[\eta^2+(1-\eta)^2\right]v_{\chi\chi} + \sigma^2\eta(1-\eta)(2\eta-1)v_{\chi\eta} + \sigma^2\eta^2(1-\eta)^2 v_{\eta\eta}.\end{aligned} \tag{39}$$

The second-order terms are the instantaneous covariance matrix implied by the diffusion in Lemma 4:

$$(d\chi)^2 = \sigma^2\left[\eta^2+(1-\eta)^2\right]dt, \qquad (d\eta)^2 = 2\sigma^2\eta^2(1-\eta)^2 dt, \qquad d\chi\, d\eta = \sigma^2\eta(1-\eta)(2\eta-1)dt.$$

Thus the mixed derivative in ((39)) has covariance coefficient $\sigma^2\eta(1-\eta)(2\eta-1)$.

The HJB in the main text is therefore

$$\min\left\{\rho v - \mathcal{L}_{\chi\eta}v - \left(e^{(1-\nu)(\chi-\chi_0)}-1\right),\, v\right\} = 0.$$

It is often more convenient in the dynamic appendices to use (u,z), where

$$z = \log\frac{\eta}{1-\eta}, \qquad u = \chi - \log H(z), \qquad H(z) = e^{z/2}+e^{-z/2}.$$

In these variables, the same generator becomes the constant-coefficient operator

$$\mathscr{L}_{uz}\varphi(u,z) = \frac{g-\mu_a-\mu_b}{2}\varphi_u + (g+\mu_b-\mu_a)\varphi_z + \frac{\sigma^2}{4}\varphi_{uu} + \sigma^2\varphi_{zz}. \tag{40}$$

Writing $\widetilde{g} = g-\mu_a$ gives the notation used in the next appendix subsection:

$$\mathscr{L}_{\widetilde{g}}\varphi = \frac{\widetilde{g}-\mu_b}{2}\varphi_u + (\widetilde{g}+\mu_b)\varphi_z + \frac{\sigma^2}{4}\varphi_{uu} + \sigma^2\varphi_{zz}.$$

Its forward adjoint is the Kolmogorov operator for the stationary density. The shifted-entry source term belongs to that population equation and is not part of the private incumbent generator.

6.4 Relative growth invariance

Corollary 1. *Suppose entry costs are measured in units of output, the rental rate is constant on the balanced growth path, and the source intensity in the KFE is either fixed or determined from the normalized stationary firm block. Then there exist functions* $v_{\widetilde{g}}$, $f_{\widetilde{g}}$, *and* $\bar{v}_E(\widetilde{g})$ *such that*

$$v_{g,\mu_a}(u,z) = v_{\widetilde{g}}(u,z), \qquad f_{g,\mu_a}(u,z) = f_{\widetilde{g}}(u,z), \qquad \bar{v}_E(g,\mu_a) = \bar{v}_E(g-\mu_a).$$

Consequently, free entry pins down $\widetilde{g} = g - \mu_a$*:*

$$c^E = \bar{v}_E(\widetilde{g}).$$

The tail exponent and the tail labor-share ratio are therefore functions of $\widetilde{g}$ *and the remaining primitives:*

$$\zeta^* = \frac{\widetilde{g}-\mu_b}{\sigma^2} - \frac{1}{\Delta} + \sqrt{\frac{(\widetilde{g}-\mu_b)^2}{\sigma^4} + \frac{1}{\Delta^2} + \frac{(\widetilde{g}+\mu_b)^2}{\sigma^4}}, \qquad \frac{\alpha}{\alpha+\beta} = \frac{1}{2} + \frac{\widetilde{g}+\mu_b}{2\sigma^2\zeta^*}.$$

The proof is a change-of-variables result. The firm problem is written in terms of factor costs relative to firm productivities, so an equal increase in g and μ_a leaves all relative state dynamics unchanged.

Proof. Let $\widetilde{g} = g - \mu_a$. Then

$$g - \mu_a - \mu_b = \widetilde{g} - \mu_b, \qquad g + \mu_b - \mu_a = \widetilde{g} + \mu_b.$$

The diffusion in (u,z) can therefore be written as

$$du = \frac{\widetilde{g}-\mu_b}{2}\,dt - \frac{\sigma}{\sqrt{2}}\,dW^u, \qquad dz = (\widetilde{g}+\mu_b)\,dt + \sqrt{2}\sigma\,dW^z,$$

and the forward generator is

$$\mathscr{L}^*_{\widetilde{g}} f = -\frac{\widetilde{g}-\mu_b}{2} f_u - (\widetilde{g}+\mu_b) f_z + \frac{\sigma^2}{4} f_{uu} + \sigma^2 f_{zz}.$$

Thus the law of motion depends on (g,μ_a) only through $\widetilde{g}$.

The normalized HJB is

$$\min\left\{\rho v - \mathscr{L}_{\widetilde{g}} v - \left(\frac{\kappa_\nu}{F} e^{(1-\nu)u} H(z)^{1-\nu} - 1\right),\ v\right\} = 0.$$

The flow payoff contains no separate g, μ_a, or R once the problem is written in the normalized coordinates. Hence, under uniqueness of the nonnegative solution to the obstacle problem, the value function and continuation set satisfy

$$v_{g,\mu_a}(u,z) = v_{\widetilde{g}}(u,z), \qquad \mathfrak{C}_{g,\mu_a} = \mathfrak{C}_{\widetilde{g}}.$$

The level of the rental rate affects the mapping from physical productivities to the normalized state, but not the normalized PDE itself. The condition used here is that the rental rate is constant on the balanced growth path, so that it does not enter the drift of the state variables.

For a given source intensity, the stationary KFE becomes

$$0 = \mathscr{L}^*_{\widetilde{g}} f(u,z) + \varepsilon_A f(u-\Delta, z), \qquad (u,z) \in \mathfrak{C}_{\widetilde{g}}.$$

The boundary condition, the imitation shift, and the normalization $\iint_{\mathfrak{C}_{\widetilde{g}}} f = 1$ contain no separate dependence on g or μ_a. Therefore, whenever the normalized KFE has a unique stationary density,

$$f_{g,\mu_a}(u,z) = f_{\widetilde{g}}(u,z).$$

If ε_A is itself determined by the normalized stationary block, the same argument applies because that block is invariant. It would fail only if ε_A were imposed as a separate function of absolute g.

Combining the HJB and KFE invariance gives the entry-value result:

$$\bar{v}_E(g,\mu_a) = \iint_{\mathfrak{C}_{\widetilde{g}}} v_{\widetilde{g}}(u+\Delta, z) f_{\widetilde{g}}(u,z)\, du\, dz \equiv \bar{v}_E(\widetilde{g}).$$

Since entry costs are measured in units of output, the free-entry condition is $c^E = \bar{v}_E(\widetilde{g})$. It pins down $\widetilde{g}$. For fixed μ_a this is an equation in g, but across values of μ_a it implies $g = \mu_a + \widetilde{g}$.

The same substitution gives the tail formulas. The tilted tail equation is

$$\omega_\zeta''(z) - m(\zeta)^2 \omega_\zeta(z) = 0,$$

where

$$m(\zeta)^2 = -\frac{\zeta^2}{4} + \frac{\widetilde{g} - \mu_b}{2\sigma^2}\zeta - \frac{\varepsilon_A}{\sigma^2}e^{-\zeta\Delta} + \frac{(\widetilde{g} + \mu_b)^2}{4\sigma^4}.$$

Solving the repeated-root system gives

$$\zeta^* = \frac{\widetilde{g} - \mu_b}{\sigma^2} - \frac{1}{\Delta} + \sqrt{\frac{(\widetilde{g} - \mu_b)^2}{\sigma^4} + \frac{1}{\Delta^2} + \frac{(\widetilde{g} + \mu_b)^2}{\sigma^4}},$$

and the tail labor-share ratio is

$$\frac{\alpha}{\alpha + \beta} = \frac{1}{2} + \frac{\widetilde{g} + \mu_b}{2\sigma^2\zeta^*}.$$

These expressions contain μ_a only through $\widetilde{g}$. Finally, the aggregate block maps $(\mu_a, \widetilde{g})$ into $g = \mu_a + \widetilde{g}$ and $R = \rho + g + \delta$. This can change aggregate ratios such as K/Y, but it does not feed back into the normalized firm-side problem under the maintained normalization. This proves the corollary.

6.5 High-entry-cost free entry

This appendix proves the high-entry-cost uniqueness result used in the free-entry proposition. Let $\mathcal{G}$ be the relevant set of candidate growth rates, and let $G_T = (\underline{g}_T, \overline{g}_T)$ be the lower-tail region. Write $q(g) = \zeta(g) - (\nu - 1)$. Assume that $q(g) > 0$ on G_T, that $q(g) \downarrow 0$ as $g \downarrow \underline{g}_T$, and that $q'(g)$ is bounded below by a positive constant near $\underline{g}_T$. Suppose that, on this tail region,

$$\bar{v}_E(g) = \frac{C(g)}{q(g)} + R(g),$$

where $C(g)$ is continuously differentiable and bounded away from zero, while $C'(g)$ and $R'(g)$ are bounded near $\underline{g}_T$. Finally, assume that $\bar{v}_E(g)$ is bounded above on any part of $\mathcal{G}$ bounded away from $\underline{g}_T$.

Proof. Differentiating the tail expansion gives

$$\bar{v}'_E(g) = \frac{C'(g)}{q(g)} - \frac{C(g)q'(g)}{q(g)^2} + R'(g).$$

As $g \downarrow \underline{g}_T$, the term $-C(g)q'(g)/q(g)^2$ dominates the remaining terms, because $C(g)$ and $q'(g)$ are bounded away from zero and the other terms are at most of order $1/q(g)$. Hence there is some $g_T \in G_T$ such that $\bar{v}'_E(g) < 0$ on $(\underline{g}_T, g_T]$. The same expansion implies $\bar{v}_E(g) \to \infty$ as $g \downarrow \underline{g}_T$. Let

$$M_T = \sup_{g \in \mathcal{G} \setminus (\underline{g}_T, g_T]} \bar{v}_E(g) < \infty$$

and define $\bar{c}^E_T = \max\{\bar{v}_E(g_T), M_T\}$. If $c^E > \bar{c}^E_T$, no solution to $c^E = \bar{v}_E(g)$ can lie outside $(\underline{g}_T, g_T]$. On that interval, existence follows from continuity and $\bar{v}_E(g) \to \infty$ at the lower endpoint, while uniqueness follows from strict monotonicity. Since $\bar{v}_E$ is strictly decreasing there, the inverse growth rate is decreasing in c^E. $\square$

6.6 The dynamic exit boundary

This appendix proves Lemma 5. Write normalized flow profits as

$$\pi(\chi) = e^{(1-\nu)(\chi-\chi_0)} - 1.$$

Then $\pi(\chi_0) = 0$, $\pi(\chi) > 0$ if and only if $\chi < \chi_0$, and π is strictly decreasing in χ.

Proof. Fix $\eta_0 \in (0,1)$ and two initial costs $\chi_1 < \chi_2$. Couple the two firms with the same Brownian motions. By Lemma 4, the law of motion for η does not depend on χ, and the drift and diffusion coefficients of χ depend on η but not on the level of χ. The coupled firms therefore have the same η_t and satisfy

$$\chi_{2,t} - \chi_{1,t} = \chi_2 - \chi_1 > 0$$

for all t up to any common stopping time. Since π is decreasing, the lower-cost firm earns weakly higher flow profits under every common stopping rule. Taking suprema over stopping times gives

$$\chi_1 < \chi_2 \quad \Longrightarrow \quad v(\chi_1, \eta_0) \geq v(\chi_2, \eta_0).$$

Thus, for each η_0, the continuation set is a lower interval in χ. With the definition of $\bar{\chi}(\eta_0)$ in the main text and the usual continuity of the obstacle solution, this gives the cutoff representation of $\mathfrak{C}$ and $\mathfrak{S}$. $\square$

It remains to compare the dynamic boundary with the static zero-profit point. If $\chi < \chi_0$, current flow profits are strictly positive. The firm can continue for a short deterministic interval and then exit, so $v(\chi, \eta_0) > 0$ and $\bar{\chi}(\eta_0) \geq \chi_0$. The inequality is strict because the χ diffusion is locally nondegenerate:

$$\frac{d\langle\chi\rangle_t}{dt} = \sigma^2 \left[\eta_t^2 + (1-\eta_t)^2\right] \geq \frac{\sigma^2}{2} > 0.$$

Starting from (χ_0, η_0), fix a small band around χ_0 and let τ be the first time χ reaches either side of the band. Exit at τ if the adverse side is reached. If the favorable side is reached, switch to the short-horizon policy used above for states with $\chi < \chi_0$. Local nondegeneracy gives the favorable event strictly positive probability. The possible losses before the band is hit are bounded by continuity of π and vanish with the width of the band, while the continuation payoff on the favorable event is strictly positive. For a sufficiently small band this stopping rule has positive expected payoff. Hence $v(\chi_0, \eta_0) > 0$. By continuity, $v(\chi, \eta_0) > 0$ for some $\chi > \chi_0$, and therefore $\bar{\chi}(\eta_0) > \chi_0$.Since η_0 was arbitrary, the result holds for every $\eta \in (0, 1)$.

6.7 Conditional far-left tail selection and the double root

This appendix records the conditional proof sketch behind the double-root selection used in the main text. The calculation has two parts with different logical status. The first part is local: in the far-left cost tail, after freezing the coefficients and conjugating the Kolmogorov equation by an exponential weight, the weighted bulk term is governed by a scalar frontier. The second part is global: it assumes that the normalized stationary equilibrium selects the largest admissible nonexpanding frontier value. Under that critical-entry principle, a unique interior maximizer gives the double-root condition.

Proposition 12. *Conditional double-root selection. Fix a candidate growth rate g, an attempted entry rate ε_A, an imitation step $\Delta > 0$, and $\sigma > 0$. Let*

$$a = \frac{g - \mu_a - \mu_b}{2}, \qquad b = g + \mu_b - \mu_a, \qquad \kappa = \frac{b}{2\sigma^2}.$$

Consider the frozen far-left forward equation

$$\partial_t f = -a f_u - b f_z + \frac{\sigma^2}{4} f_{uu} + \sigma^2 f_{zz} + \varepsilon_A f(u - \Delta, z).$$

Suppose the finite-window residuals described below are negligible along a sequence of cost-tail windows, and suppose the normalized stationary problem satisfies the conditional critical-entry principle

$$\bar{\varepsilon}_A = \max_{\zeta \in \mathcal{D}} \varepsilon_A^{\mathrm{fr}}(\zeta),$$

where the admissible set $\mathcal{D}$ includes the endpoint, cutoff, exit-separation, full-model residual, and nondegenerate-mass conditions listed below. If the maximum exists, is unique, and is attained at an interior point ζ^*, then

$$m(\zeta^*; \bar{\varepsilon}_A)^2 = 0, \qquad \partial_\zeta m(\zeta^*; \bar{\varepsilon}_A)^2 = 0.$$

Moreover, the corresponding local far-left tail in economic variables is

$$p(\chi,\eta) \propto e^{\zeta^*\chi}\eta^{\alpha^*-1}(1-\eta)^{\beta^*-1}, \qquad \alpha^* = \frac{\zeta^*}{2}+\kappa, \qquad \beta^* = \frac{\zeta^*}{2}-\kappa.$$

This last expression is a local asymptotic tail formula, not a global formula for the stationary density on the whole continuation region.

Proof sketch. The proof is organized around smooth cost-tail windows. Write

$$z = \log\frac{\eta}{1-\eta}, \qquad \chi = u + \ell(z), \qquad \ell(z) = -\frac{1}{2}\log(\eta(1-\eta)).$$

Equivalently, $u = \chi + \frac{1}{2}\log(\eta(1-\eta))$. The derivative bounds

$$\ell'(z) = \frac{1}{2}\tanh(z/2), \qquad \ell''(z) = \frac{1}{4\cosh^2(z/2)}, \qquad |\ell'(z)| \le \frac{1}{2}$$

show that cost-level curves are uniformly Lipschitz in (u,z). This matters because the far-left region is a cost-tail region, not a horizontal u-box: large $|z|$ changes cost through $\ell(z)$.

Let

$$\Psi_{R,W,L}(u,z) = \rho_{R,W}(u+\ell(z))\theta_L(z) = \rho_{R,W}(\chi)\theta_L(z),$$

where $\rho_{R,W}$ is supported on a cost slab $R-W \le \chi \le R$ and θ_L cuts off the two z-tails. The artificial faces are the two cost faces, the two z-tail faces, and their smoothed overlaps. If the exit frontier satisfies $\bar{\chi}(\eta) \ge \chi_0$ and

$$R \le \chi_0 - \delta,$$

then $\operatorname{supp}\Psi_{R,W,L}$ is separated from the absorbing exit boundary. If the comparison also follows children created by the imitation shift, the stronger condition

$$R + \Delta \le \chi_0 - \delta$$

keeps the shifted states inside the separated continuation region. Thus exit is not ignored; it is absent only by support separation.

For a weight exponent ζ, set

$$h_\zeta(t,u,z) = e^{-\zeta u-\kappa z}f(t,u,z), \qquad f = e^{\zeta u+\kappa z}h_\zeta.$$

The choice $\kappa = b/(2\sigma^2)$ removes the first derivative in z from the conjugated operator. The frozen equation becomes

$$\partial_t h_\zeta = \frac{\sigma^2}{4} h_{uu} + \sigma^2 h_{zz} + \left(\frac{\sigma^2 \zeta}{2} - a \right) h_u + C(\zeta) h + \varepsilon_A e^{-\zeta \Delta} h(t, u - \Delta, z),$$

where

$$C(\zeta) = \frac{\sigma^2 \zeta^2}{4} - a\zeta - \frac{b^2}{4\sigma^2}.$$

The associated weighted flux is

$$\mathcal{F}_\zeta = \left(\frac{\sigma^2}{4} h_u + \left(\frac{\sigma^2 \zeta}{2} - a \right) h,\ \sigma^2 h_z \right).$$

Multiplying by $\Psi_{R,W,L}$, integrating by parts, and separating the nonlocal imitation shift gives the finite-window identity

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} I_{R,W,L,\zeta} = {} & s(\zeta; \varepsilon_A) I_{R,W,L,\zeta} + \mathcal{B}^{\chi}_{R,W,L,\zeta} + \mathcal{R}^{z}_{R,W,L,\zeta} + \mathcal{R}^{\Delta}_{R,W,L,\zeta} \\ & + \mathcal{R}^{\mathrm{exit}}_{R,W,L,\zeta} + \mathcal{R}^{\mathrm{model}}_{R,W,L,\zeta}, \end{aligned}$$

with

$$I_{R,W,L,\zeta} = \int \Psi_{R,W,L}(u, z) h_\zeta(t, u, z) \, \mathrm{d}u \, \mathrm{d}z.$$

The bulk coefficient is

$$s(\zeta; \varepsilon_A) = C(\zeta) + \varepsilon_A e^{-\zeta \Delta} = -\sigma^2 m(\zeta; \varepsilon_A)^2,$$

where

$$m(\zeta; \varepsilon_A)^2 = -\frac{\zeta^2}{4} - \frac{\zeta}{2\sigma^2}(\mu_a + \mu_b - g) - \frac{\varepsilon_A}{\sigma^2} e^{-\zeta \Delta} + \frac{(g + \mu_b - \mu_a)^2}{4\sigma^4}.$$

The remaining terms in the finite-window identity are not discarded. The cost-face flux is supported where $\rho'_{R,W} \neq 0$; the z-tail term is supported where $\theta'_L \neq 0$; the imitation-shift remainder is supported on the cost strips created by replacing $\rho_{R,W}(\chi + \Delta)$ by $\rho_{R,W}(\chi)$; the exit term is zero only under the exit-separation condition above; and the full-model term collects coefficient/freezing, coordinate, source, survival/obstacle, normalization, endpoint, and stationarity residuals.

The limiting regime requires these named residuals to be negligible relative to the weighted mass $I_{R,W,L,\zeta}$. A convenient order is to control the z-tails as $L \to \infty$, move the cost window into the far-left region with $R \to -\infty$ while maintaining the support separations, and then widen the cost slab with $W \to \infty$. The artificial cost-face and shift-strip terms require local slab bounds or Harnack-type estimates and nonconcentration of the weighted mass at the artificial faces. Endpoint admissibility also requires more than the formal beta inequalities: actual z-tail and endpoint flux estimates must remove the θ_L cutoff.

Under these residual and endpoint hypotheses, the finite-window identity reduces along the chosen sequence to

$$\frac{\mathrm{d}}{\mathrm{d}t} I_{R,W,L,\zeta} = s(\zeta; \varepsilon_A) I_{R,W,L,\zeta} + o(I_{R,W,L,\zeta}).$$

The local frontier is the equality case $s(\zeta; \varepsilon_A) = 0$, equivalently $m(\zeta; \varepsilon_A)^2 = 0$. Solving for ε_A gives

$$\varepsilon_A^{\mathrm{fr}}(\zeta) = \sigma^2 e^{\zeta\Delta} \left[-\frac{\zeta^2}{4} + \frac{g - \mu_a - \mu_b}{2\sigma^2}\zeta + \frac{(g + \mu_b - \mu_a)^2}{4\sigma^4} \right].$$

This is still local algebra. It identifies, for each admissible ζ, the largest attempted entry rate that is locally marginal for the corresponding weighted tail. It does not by itself prove which frontier point the normalized stationary economy selects.

The selection step is therefore stated as the critical-entry principle in the proposition. If the admissible frontier has a unique interior maximizer ζ^*, then $\bar{\varepsilon}_A = \varepsilon_A^{\mathrm{fr}}(\zeta^*)$, so

$$m(\zeta^*; \bar{\varepsilon}_A)^2 = 0.$$

Let $F(\zeta, \varepsilon_A) = m(\zeta; \varepsilon_A)^2$. Along the frontier,

$$F(\zeta, \varepsilon_A^{\mathrm{fr}}(\zeta)) = 0.$$

Differentiating gives

$$F_\zeta(\zeta, \varepsilon_A^{\mathrm{fr}}(\zeta)) + F_{\varepsilon_A}(\zeta, \varepsilon_A^{\mathrm{fr}}(\zeta))(\varepsilon_A^{\mathrm{fr}})'(\zeta) = 0,$$

where $F_{\varepsilon_A} = -e^{-\zeta\Delta}/\sigma^2 \neq 0$. At an interior maximum, $(\varepsilon_A^{\mathrm{fr}})'(\zeta^*) = 0$, hence

$$\partial_\zeta m(\zeta^*; \bar{\varepsilon}_A)^2 = 0.$$

This is the double-root implication. If the maximizer lies on the boundary of the admissible set, the derivative condition must be replaced by the appropriate one-sided or Kuhn–Tucker condition.

It remains only to translate the selected local tail back to economic variables. The selected tilted local profile is

$$f(u, z) \asymp C \exp\{\zeta^* u + \kappa z\}.$$

The Jacobian satisfies

$$\left|\frac{\partial(\chi,\eta)}{\partial(u,z)}\right| = \eta(1-\eta), \qquad p(\chi,\eta) = \frac{f(u,z)}{\eta(1-\eta)}.$$

Using $u = \chi + \frac{1}{2}\log(\eta(1-\eta))$ and $z = \log(\eta/(1-\eta))$,

$$\begin{aligned} e^{\zeta^* u + \kappa z} &= \exp\left\{\zeta^*\chi + \frac{\zeta^*}{2}\log(\eta(1-\eta)) + \kappa \log\frac{\eta}{1-\eta}\right\} \\ &= e^{\zeta^*\chi}\eta^{\zeta^*/2+\kappa}(1-\eta)^{\zeta^*/2-\kappa}. \end{aligned}$$

Dividing by $\eta(1-\eta)$ gives the displayed translated-tail formula. Thus larger positive ζ^* means a thinner far-left cost tail as $\chi \to -\infty$, while α^* and β^* describe the local endpoint powers near $\eta = 0$ and $\eta = 1$.

Caveat. The proposition should not be read as an unconditional compact-support selection theorem. The local weighted algebra, the frontier formula, the interior-maximizer calculus, and the (u, z) to (χ, η) translation are the established parts of the argument. A full theorem would still require uniform endpoint and z-tail estimates for the actual stationary density, local slab or Harnack-type estimates for the weighted density, full-model residual bounds, a positivity and nondegenerate-mass result for the tested windows, a normalized-KFE proof of the critical-entry principle, and either interiority of the maximizing exponent or a boundary KKT replacement.

6.8 Correlated and unequal productivity shocks in the dynamic model

This appendix derives the tail approximation when labor- and capital-augmenting productivity shocks have different volatilities and instantaneous correlation. Suppose

$$d\log a_t = \mu_a dt + \sigma_a dW_t^a, \qquad d\log b_t = \mu_b dt + \sigma_b dW_t^b, \qquad d\langle W^a, W^b\rangle_t = \rho dt.$$

Let

$$x_t = \log(w_t/a_t), \qquad y_t = \log(R/b_t), \qquad u_t = \frac{x_t + y_t}{2}, \qquad z_t = x_t - y_t.$$

On a balanced growth path,

$$du_t = \frac{g - \mu_a - \mu_b}{2}dt - \frac{1}{2}\left(\sigma_a dW_t^a + \sigma_b dW_t^b\right), \qquad dz_t = (g - \mu_a + \mu_b)dt - \sigma_a dW_t^a + \sigma_b dW_t^b.$$

The instantaneous variance-covariance terms in (u, z) are

$$Q_u = \frac{\sigma_a^2 + \sigma_b^2 + 2\rho\sigma_a\sigma_b}{4}, \qquad Q_z = \sigma_a^2 + \sigma_b^2 - 2\rho\sigma_a\sigma_b, \qquad Q_{uz} = \frac{\sigma_a^2 - \sigma_b^2}{2}.$$

Writing

$$m_u = \frac{g - \mu_a - \mu_b}{2}, \qquad m_z = g - \mu_a + \mu_b,$$

the constant-coefficient part of the forward equation in the tail is

$$\mathcal{L}^* f = -m_u f_u - m_z f_z + \frac{Q_u}{2} f_{uu} + Q_{uz} f_{uz} + \frac{Q_z}{2} f_{zz}.$$

For a candidate tail exponent ζ, write $f(u, z) = e^{\zeta u}\varphi_\zeta(z)$. Substitution gives

$$\frac{Q_z}{2}\varphi_\zeta''(z) + (Q_{uz}\zeta - m_z)\,\varphi_\zeta'(z) + \left(\frac{Q_u}{2}\zeta^2 - m_u\zeta + \varepsilon_A e^{-\zeta\Delta}\right)\varphi_\zeta(z) = 0.$$

The first derivative is removed by the exponential tilt

$$\varphi_\zeta(z) = \exp\{\theta_\zeta z\}\omega_\zeta(z), \qquad \theta_\zeta = \frac{m_z - Q_{uz}\zeta}{Q_z}.$$

On the selected linear branch,ω_ζ is constant, and therefore

$$f_{u,z}(u, z) \propto \exp\{\zeta u + \theta_\zeta z\}.$$

The implied beta parameters are

$$\alpha = \frac{\zeta}{2} + \theta_\zeta, \qquad \beta = \frac{\zeta}{2} - \theta_\zeta.$$

Thus unequal volatilities do not curve the tail iso-density loci. They change their tilt through Q_{uz} and Q_z, while the interior-tail aggregation remains Cobb-Douglas whenever $Q_z > 0$ and the implied α and β are positive.

Holding the selected tail exponent fixed at $\bar{\zeta}$, define

$$\Gamma = \frac{\sigma_b}{\sqrt{Q_z}} = \frac{1}{\sqrt{\frac{\sigma_a^2}{\sigma_b^2} + 1 - 2\rho\frac{\sigma_a}{\sigma_b}}}, \qquad \Lambda = \sigma_b^2\left(\Gamma^{-1} + \rho\frac{\sigma_a}{\sigma_b} - 1\right).$$

The repeated-root calculation then gives, to first order in μ_b and $1/\Delta$,

$$g \approx \mu_a + (\Gamma^{-1} - 1)\mu_b + \Lambda\left(\bar{\zeta} + \frac{1}{\Delta}\right).$$

The associated tail labor share is

$$\frac{\alpha}{\alpha+\beta} \approx \Gamma + \Gamma\frac{\mu_b}{\sigma_b^2\bar{\zeta}} + \Gamma^2\frac{\Lambda}{\sigma_b^2\bar{\zeta}\Delta}.$$

When $\sigma_a = \sigma_b = \sigma$ and $\rho = 0$, one has $\Gamma = 1/\sqrt{2}$ and $\Lambda = \sigma^2(\sqrt{2}-1)$, which recovers Proposition 10.

6.9 CES production for individual firms in the dynamic model

Let individual firms have CES production with elasticity $0 \leq \epsilon < 1$, and write

$$x = \frac{w}{a}, \qquad y = \frac{R}{b}, \qquad s = 1-\epsilon > 0, \qquad c_\epsilon(x,y) = (x^s + y^s)^{1/s}.$$

Define the same normalized states as in the baseline dynamic model,

$$u = \frac{1}{2}(\log x + \log y), \qquad z = \log x - \log y.$$

Then

$$c_\epsilon(x,y) = e^u H_\epsilon(z), \qquad H_\epsilon(z) = \left(e^{sz/2} + e^{-sz/2}\right)^{1/s}.$$

The firm labor share is

$$\eta_\epsilon(z) = \frac{\partial \log c_\epsilon}{\partial \log w} = \frac{x^s}{x^s+y^s} = \frac{1}{1+e^{-sz}}, \qquad z = \frac{1}{s}\log\frac{\eta_\epsilon}{1-\eta_\epsilon}.$$

Thus CES production changes the transformation from z to the firm labor share, but not the law of motion for (u,z).

Since the diffusion in (u,z) is unchanged, the low-cost tail equation is also unchanged conditional on a growth rate and a tail exponent. Hence the same exponential tail takes the form

$$f_{u,z}(u,z) \propto \exp\{\zeta u + \gamma z\}, \qquad \gamma = \frac{g + \mu_b - \mu_a}{2\sigma^2}.$$

Equivalently,

$$\alpha = \frac{\zeta}{2} + \gamma, \qquad \beta = \frac{\zeta}{2} - \gamma.$$

The tail iso-density curves in $(\log a, \log b)$ remain linear. Firm-level CES substitution bends iso-cost curves and the exit boundary, but not the local tail geometry that generates the Cobb-Douglas aggregation result.

It remains to translate the tail density into labor-share coordinates. Let $\chi = \log c_\epsilon =$

$u + \log H_\epsilon(z)$. Since

$$H_\epsilon(z) = [\eta_\epsilon(1-\eta_\epsilon)]^{-1/(2s)},$$

the change of variables from (u, z) to (χ, η_ϵ) gives, up to a normalizing constant,

$$f_{\chi,\eta}(\chi, \eta) \propto e^{\zeta\chi}\eta^{\alpha/s-1}(1-\eta)^{\beta/s-1}.$$

Therefore, conditional on being in the low-cost tail,

$$\eta_\epsilon \sim \text{Beta}\left(\frac{\alpha}{1-\epsilon}, \frac{\beta}{1-\epsilon}\right).$$

The mean is unchanged:

$$\mathbb{E}[\eta_\epsilon] = \frac{\alpha}{\alpha+\beta} = \frac{1}{2} + \frac{g+\mu_b-\mu_a}{2\sigma^2\zeta}.$$

Consequently, holding $\bar{\zeta}$ fixed, CES production for individual firms preserves the high-entry-cost approximations in Proposition 10:

$$g \approx \mu_a + \left(\sqrt{2}-1\right)\left\{\mu_b + \sigma^2\bar{\zeta} + \frac{\sigma^2}{\Delta}\right\},$$

and

$$\frac{\alpha}{\alpha+\beta} \approx \frac{1}{\sqrt{2}} + \frac{\mu_b}{\sqrt{2}\sigma^2\bar{\zeta}} + \frac{\sqrt{2}-1}{2\bar{\zeta}}\frac{1}{\Delta}.$$

The effect of ϵ is instead to rescale the beta parameters by $1/(1-\epsilon)$, and hence to change the dispersion of firm labor shares around the same tail mean.

6.10 Aggregate CES-1/2 diffusion in the dynamic model

This appendix derives the state-dependent drift used in the dynamic aggregate CES-1/2 extension. The derivation is an interior tail calculation. It shows that the proposed process supports the CES-1/2 density away from the absorbing exit boundary; as in the baseline tail calculation, the boundary itself generates a local correction near exit.

At date t, take the productivity density to be

$$\psi_t(a, b) \propto \frac{a^{-1/2}b^{-1/2}}{\left(\lambda_{L,t}^2 a + \lambda_K^2 b\right)^{m+1}}.$$

Define the same normalized states as in the baseline dynamic model:

$$u = \frac{1}{2}\left(\log \frac{w_t}{a} + \log \frac{R}{b}\right), \qquad z = \log \frac{w_t/a}{R/b}.$$

Then

$$a = w_t e^{-u-z/2}, \qquad b = Re^{-u+z/2}.$$

Using $\lambda_{L,t}^2 w_t = \Lambda_L$ and $\lambda_K^2 R = \Lambda_K$, the density in (u, z) becomes

$$f^*(u, z) \propto e^{mu}\varphi(z), \qquad \varphi(z) = \left[\Lambda_L e^{-z/2} + \Lambda_K e^{z/2}\right]^{-\theta}, \qquad \theta = m + 1.$$

The focal point is

$$\bar{z} = \log \frac{\Lambda_L}{\Lambda_K}, \qquad \varphi'(\bar{z}) = 0.$$

The proposed interior diffusion is

$$du_t = \frac{g}{2}\,dt - \frac{\sigma}{\sqrt{2}}\,dW_t^u, \qquad dz_t = [g + b(z_t)]\,dt + \sqrt{2}\sigma\,dW_t^z.$$

Away from the absorbing boundary, the stationary KFE is

$$0 = -\frac{g}{2}f_u - \partial_z\left([g + b(z)]\,f\right) + \frac{\sigma^2}{4}f_{uu} + \sigma^2 f_{zz} + \varepsilon_A f(u - \Delta, z).$$

Substitute $f(u, z) = e^{mu}\varphi(z)$. Then

$$f_u = mf, \qquad f_{uu} = m^2 f, \qquad f(u - \Delta, z) = e^{-m\Delta}f(u, z).$$

Collect the terms coming from the u dimension and imitation entry as

$$\mathcal{A}_m = \frac{\sigma^2 m^2}{4} - \frac{gm}{2} + \varepsilon_A e^{-m\Delta}.$$

The KFE reduces to

$$\partial_z\left([g + b(z)]\,\varphi(z) - \sigma^2\varphi'(z)\right) = \mathcal{A}_m\varphi(z).$$

Integrating from $\bar{z}$ to z, imposing $b(\bar{z}) = 0$, and using $\varphi'(\bar{z}) = 0$ gives

$$[g + b(z)]\,\varphi(z) - \sigma^2\varphi'(z) = g\varphi(\bar{z}) + \mathcal{A}_m\int_{\bar{z}}^{z}\varphi(s)\,ds.$$

Solving for $b(z)$,

$$b(z) = \sigma^2 \partial_z \log \varphi(z) + g \left[\frac{\varphi(\bar{z})}{\varphi(z)} - 1 \right] + \mathcal{A}_m \frac{\int_{\bar{z}}^{z} \varphi(s)\, ds}{\varphi(z)}.$$

This is the compact form of the added drift.

To write the drift explicitly, let $\hat{z} = z - \bar{z}$. Since

$$\Lambda_L e^{-z/2} + \Lambda_K e^{z/2} = 2\sqrt{\Lambda_L \Lambda_K} \cosh\left(\frac{\hat{z}}{2}\right),$$

we have

$$\varphi(z) = C \cosh^{-\theta}\left(\frac{\hat{z}}{2}\right), \qquad \partial_z \log \varphi(z) = -\frac{\theta}{2} \tanh\left(\frac{\hat{z}}{2}\right),$$

where C is a positive constant. Therefore

$$\begin{aligned} b(\bar{z} + \hat{z}) = & -\frac{\sigma^2 \theta}{2} \tanh\left(\frac{\hat{z}}{2}\right) + g \left[\cosh^{\theta}\left(\frac{\hat{z}}{2}\right) - 1 \right] \\ & + \mathcal{A}_m \cosh^{\theta}\left(\frac{\hat{z}}{2}\right) \int_0^{\hat{z}} \cosh^{-\theta}\left(\frac{s}{2}\right) ds. \end{aligned}$$

By construction, $b(\bar{z}) = 0$. Differentiating the integrated equation above recovers the reduced KFE. Hence $f^*(u, z) = e^{mu}\varphi(z)$ solves the interior stationary equation, and the expression gives an explicit proof-of-concept diffusion that supports the CES-1/2 density in the balanced-growth interior.

6.11 The Leontief benchmark

Lemma 13. *Suppose individual unit costs are* $c(a, b; w, R) = w/a + R/b$ *and the high-productivity density is proportional to*

$$\frac{a^{-1/2} b^{-1/2}}{(\lambda_L^2 a + \lambda_K^2 b)^{m+1}}, \qquad m > \nu - 1.$$

On an interior region in which the zero-profit active set is not truncated by the support bounds, aggregate relative factor demands satisfy

$$\frac{L}{K} = \frac{\lambda_L}{\lambda_K} \left(\frac{w}{R}\right)^{-1/2}.$$

Thus the aggregate substitution margin has elasticity 1/2.

Proof. Let $\eta = (w/a)/(w/a + R/b)$ denote the firm labor share out of variable cost and set

$$s = \frac{1}{P}\left(\frac{w}{a} + \frac{R}{b}\right), \qquad t = \frac{\eta}{1-\eta} = \frac{w/a}{R/b}.$$

Then $a = w(1+t)/(Pst)$, $b = R(1+t)/(Ps)$, and

$$dadb = \frac{wR(1+t)^2}{P^2 s^3 t^2}\, dsdt.$$

The interior assumption implies that the transformed active-set limits are independent of (w, R, P). The s-integral is common to both factor-payment terms and is finite because $m > \nu - 1$. Therefore

$$\frac{wL}{RK} = \frac{\int_0^\infty \frac{t^{m+1/2}}{(1+t)^{m+1}(\lambda_L^2 w + \lambda_K^2 Rt)^{m+1}}\, dt}{\int_0^\infty \frac{t^{m-1/2}}{(1+t)^{m+1}(\lambda_L^2 w + \lambda_K^2 Rt)^{m+1}}\, dt}.$$

In the numerator, use $\hat{t} = (\lambda_L^2 w)/(\lambda_K^2 Rt)$. This gives

$$\int_0^\infty \frac{t^{m+1/2}}{(1+t)^{m+1}(\lambda_L^2 w + \lambda_K^2 Rt)^{m+1}}\, dt = \left(\frac{\lambda_L^2 w}{\lambda_K^2 R}\right)^{1/2} \int_0^\infty \frac{\hat{t}^{m-1/2}}{(1+\hat{t})^{m+1}(\lambda_L^2 w + \lambda_K^2 R\hat{t})^{m+1}}\, d\hat{t}.$$

The remaining integral is the denominator. Hence

$$\frac{wL}{RK} = \left(\frac{\lambda_L^2 w}{\lambda_K^2 R}\right)^{1/2}, \qquad \frac{L}{K} = \frac{\lambda_L}{\lambda_K}\left(\frac{w}{R}\right)^{-1/2}.$$

□

6.12 Power-law revenue tail, labor share, and firm capital-labor ratio

This appendix proves the proposition by changing variables from firm types to revenue and factor-share coordinates. The same argument yields the Pareto revenue tail, the Beta labor-share distribution, and the beta-logistic distribution of the firm capital-labor ratio.

To derive the revenue tail, define

$$u = \alpha \log \tilde{a} + \beta \log \tilde{b}, \qquad v = \log \tilde{a} - \log \tilde{b}.$$

Because the unit cost is homogeneous of degree one,

$$c(1/\tilde{a}, 1/\tilde{b}) = e^{-u/(\alpha+\beta)} C(v)$$

for some function$C(v)$. Hence

$$R_i \propto e^{\frac{\nu-1}{\alpha+\beta}u} C(v)^{1-\nu}.$$

Under$\psi(x) \propto x^{-\tau}$, the density in$(u, v)$ coordinates is proportional to$e^{-\tau u}$. Therefore,$R_i > z$ is equivalent to

$$u > \frac{\alpha+\beta}{\nu-1} \log z + (\alpha+\beta) \log C(v) + \text{const.}$$

and thus

$$\Pr(R_i > z \,|\, active) \propto \int \int_{u > \bar{u}(v,z)} e^{-\tau u} \, du \, dv,$$

where$\bar{u}(v, z)$ denotes the threshold above. Evaluating the$u$ integral gives

$$\Pr(R_i > z \,|\, active) \sim \mathcal{C} z^{-\frac{(\alpha+\beta)\tau}{\nu-1}}$$

for some finite constant$\mathcal{C} > 0$. Thus the revenue tail is Pareto with exponent$\frac{(\alpha+\beta)\tau}{\nu-1}$, which reduces to$\frac{\alpha+\beta}{\nu-1}$ under independent Pareto.

For the labor share distribution, define

$$q := \frac{w/a}{R/b}, \qquad s := c(1/\tilde{a}, 1/\tilde{b}),$$

so that normalized revenue is proportional to$s^{1-\nu}$ and firm size is a monotone transformation of$s$.

On the homogeneous interior,$s = (1/\tilde{b})c(q, 1)$ and therefore

$$\frac{1}{\tilde{b}} = \frac{s}{c(q,1)}, \qquad \frac{1}{\tilde{a}} = \frac{qs}{c(q,1)}.$$

For the balanced class$\psi(x) \propto x^{-\tau}$, the same change of variables gives

$$f_{q,s}(q, s) \propto q^{\alpha\tau-1} c(q, 1)^{-(\alpha+\beta)\tau} s^{(\alpha+\beta)\tau-1}.$$

Thus the joint density factorizes into a function of$q$ times a function of$s$. Hence$q$is independent of$s$, and therefore the labor share is independent of firm size. For the CES production function

$$y = \left[(al)^{\frac{\epsilon-1}{\epsilon}} + (bk)^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{\epsilon}{\epsilon-1}},$$

the dual cost satisfies

$$c(q, 1) = \left(q^{1-\epsilon} + 1 \right)^{\frac{1}{1-\epsilon}},$$

and the labor share is

$$\eta(q) = \frac{qc_1(q,1)}{c(q,1)} = \frac{q^{1-\epsilon}}{1+q^{1-\epsilon}}.$$

Equivalently,

$$q = \left(\frac{\eta}{1-\eta}\right)^{\frac{1}{1-\epsilon}}.$$

Substituting this into the$q$ density and applying the Jacobian gives

$$f_\eta(\eta) \propto \eta^{\frac{\alpha\tau}{1-\epsilon}-1}(1-\eta)^{\frac{\beta\tau}{1-\epsilon}-1}, \qquad 0 < \eta < 1.$$

Thus

$$\eta \sim \text{Beta}\left(\frac{\alpha\tau}{1-\epsilon}, \frac{\beta\tau}{1-\epsilon}\right),$$

and becauseη is independent of$s$ it is also independent of firm revenue size. In the independent Pareto benchmark$\tau = 1$, this reduces to

$$\eta \sim \text{Beta}\left(\frac{\alpha}{1-\epsilon}, \frac{\beta}{1-\epsilon}\right).$$

Finally, the cost shares satisfy

$$\eta_i = \frac{wl_i}{wl_i + Rk_i},$$

so

$$\frac{k_i}{l_i} = \frac{w}{R}\frac{1-\eta_i}{\eta_i}.$$

Hence

$$x := \log(k_i/l_i) - \log(w/R) = \log\frac{1-\eta_i}{\eta_i}.$$

Becauseη_i is independent of$s$, the same is true for$x$. Its density follows from the change of variables

$$\eta_i = \frac{1}{1+e^x}, \qquad \left|\frac{d\eta_i}{dx}\right| = \frac{e^x}{(1+e^x)^2}.$$

Substituting into the Beta density gives

$$f_x(x) = \frac{1}{\mathrm{B}\left(\frac{\alpha\tau}{1-\epsilon}, \frac{\beta\tau}{1-\epsilon}\right)} \frac{\exp\left(\frac{\beta\tau}{1-\epsilon}x\right)}{(1+e^x)^{\frac{(\alpha+\beta)\tau}{1-\epsilon}}}, \qquad x \in \mathbb{R},$$

which is a beta-logistic density.

6.13 Beyond exact independence

This appendix proves Proposition 3. The first part gives a tractable Leontief example in which η and size are not independent. The second part shows that the conditional mean $\mathbb{E}(\eta \mid s)$ remains equal to $\alpha/(\alpha+\beta)$ throughout the balanced interior class.

Under CES production, $c(x,y) = (x^{1-\epsilon}+y^{1-\epsilon})^{\frac{1}{1-\epsilon}}$, write $x = w/a$, $y = R/b$, $q = x/y$, $s = c(x,y)/P$, and let $C(q) = c(q,1)$. Then $y = s/C(q)$ and $x = qs/C(q)$. The labor share is

$$\eta(q) = \frac{x^{1-\epsilon}}{x^{1-\epsilon}+y^{1-\epsilon}} = \frac{q^{1-\epsilon}}{1+q^{1-\epsilon}},$$

so

$$q = \left(\frac{\eta}{1-\eta}\right)^{\frac{1}{1-\epsilon}}.$$

For the balanced class $g(a,b) = \frac{1}{ab}\psi(a^{\alpha}b^{\beta})$, the joint density kernel in (q,s) is

$$f_{q,s}(q,s) = \frac{1}{qs}\psi\Big(\kappa_P^{-(\alpha+\beta)}q^{-\alpha}C(q)^{\alpha+\beta}s^{-(\alpha+\beta)}\Big).$$

If $\psi(z) = Dz^{-\tau}e^{-\lambda/z}$, this becomes

$$f_{q,s}(q,s) \propto q^{\alpha\tau-1}C(q)^{-(\alpha+\beta)\tau}s^{(\alpha+\beta)\tau-1}\exp\left\{-\bar{\lambda}s^{\alpha+\beta}q^{\alpha}C(q)^{-(\alpha+\beta)}\right\}.$$

Using $q = (\eta/(1-\eta))^{1/(1-\epsilon)}$ and $|dq/d\eta| = q/\left[(1-\epsilon)\eta(1-\eta)\right]$, this becomes

$$f_{\eta,s}(e,s) \propto s^{(\alpha+\beta)\tau-1}e^{\frac{\alpha\tau}{1-\epsilon}-1}(1-e)^{\frac{\beta\tau}{1-\epsilon}-1}\exp\left\{-\bar{\lambda}s^{\alpha+\beta}e^{\frac{\alpha}{1-\epsilon}}(1-e)^{\frac{\beta}{1-\epsilon}}\right\}.$$

where $\bar{\lambda} = \lambda\kappa_P^{\alpha+\beta}$ is constant. Hence, conditional on s, the density of η, and therefore the density conditional on firm size, is an exponentially tilted $\mathrm{Beta}\big(\frac{\alpha\tau}{1-\epsilon}, \frac{\beta\tau}{1-\epsilon}\big)$ law. When $\lambda = 0$ the tilt disappears and the power-law benchmark is recovered.

For the broader statement, keep CES production and the balanced interior class but allow an arbitrary admissible ψ. Write $q = (w/a)/(R/b)$, $s = c(w/a, R/b)$, and $C(q) = c(q,1)$. Then

$$f_{q,s}(q,s) = \frac{1}{qs}\psi\big(w^{\alpha}R^{\beta}q^{-\alpha}C(q)^{\alpha+\beta}s^{-(\alpha+\beta)}\big).$$

Let $A(q) = q^{-\alpha}C(q)^{\alpha+\beta}$ and $\eta(q) = qC'(q)/C(q)$. The identity

$$\frac{d}{dq}\log A(q) = \frac{(\alpha+\beta)\eta(q)-\alpha}{q}$$

implies that

$$(\alpha+\beta)\mathbb{E}(\eta \mid s)-\alpha = \frac{\int \psi\big(w^{\alpha}R^{\beta}A(q)s^{-(\alpha+\beta)}\big)\ d\log A(q)}{\int q^{-1}\psi(w^{\alpha}R^{\beta}A(q)s^{-(\alpha+\beta)})\ dq}.$$

On the homogeneous interior,$A(q)$ tends to infinity at both endpoints and has a unique minimum, so the two angular branches cancel in the numerator. Therefore

$$\mathbb{E}(\eta \mid s)=\frac{\alpha}{\alpha+\beta}$$

for every interior cost level$s$. This pins down the first conditional moment without implying independence ofη and$s$.

6.14 Heterogeneous markups under radial dependence

Write the conditional productivity density as

$$g(a,b \mid \gamma)=\frac{1}{ab}\psi_{\gamma}(a^{\alpha}b^{\beta}), \tag{41}$$

with the same α and β for every γ. This is weaker than independence: it allows γ to be correlated with the radial productivity index $a^{\alpha}b^{\beta}$, but not with the relative productivity direction a/b.

I use regular Kimball demand to refer to the following sufficient restrictions. For each relevant γ, $\Psi(0,\gamma)=0$, $\Psi_x>0$, $\Psi_{xx}<0$, and the marginal-revenue index $x\mapsto \Psi_x(x,\gamma)+x\Psi_{xx}(x,\gamma)$ is positive and strictly decreasing on the range used in equilibrium. The induced operating-surplus function $R(s,\gamma)$ is continuous and strictly decreasing in normalized cost s, the zero-profit equation has a finite positive cutoff $\bar{s}_{\mathcal{A}}(\gamma)$ for almost every active γ, and the active region is interior with finite price, factor-demand, and mass integrals.

Under regular Kimball demand, firm behavior depends on type only through normalized cost and γ. In cost-adjusted coordinates

$$\tilde{a}\equiv\frac{a\lambda}{w}, \qquad \tilde{b}\equiv\frac{b\lambda}{R},$$

normalized cost is $s(\tilde{a},\tilde{b})=c(1/\tilde{a},1/\tilde{b})$, so each γ class has active set

$$\Omega(w,R,\lambda)=\left\{(a,b,\gamma):\frac{c(w/a,R/b)}{\lambda}\le \bar{s}_{\mathcal{A}}(\gamma)\right\}. \tag{42}$$

After the change of variables,

$$g(a, b \mid \gamma)\ \mathrm{d}a\ \mathrm{d}b = \frac{1}{\tilde{a}\tilde{b}}\psi_\gamma\Big(Q\tilde{a}^\alpha\tilde{b}^\beta\Big)\ \mathrm{d}\tilde{a}\ \mathrm{d}\tilde{b}, \qquad Q \equiv \frac{w^\alpha R^\beta}{\lambda^{\alpha+\beta}}.$$

Hence all factor-price dependence enters through the single scalar Q. If the corresponding scalar equilibrium system has an interior solution $(\mathcal{A}_*, Q_*)$, then

$$\lambda = Q_*^{-1/(\alpha+\beta)} w^{\alpha/(\alpha+\beta)} R^{\beta/(\alpha+\beta)}, \qquad P = \mathcal{A}_* Q_*^{-1/(\alpha+\beta)} w^{\alpha/(\alpha+\beta)} R^{\beta/(\alpha+\beta)}. \tag{43}$$

Because γ changes only radial weights, the same balanced-angular identity as in the baseline applies inside each γ block. With $\mathcal{V}_*$ denoting the aggregate inverse-markup term, this gives

$$\frac{wL}{PY} = \frac{\alpha}{\alpha+\beta}\mathcal{V}_*, \qquad \frac{RK}{PY} = \frac{\beta}{\alpha+\beta}\mathcal{V}_*. \tag{44}$$

and therefore $wL/RK = \alpha/\beta$. Combining (43) and (44) yields

$$Y(L, K) = A_* L^{\alpha/(\alpha+\beta)} K^{\beta/(\alpha+\beta)}.$$

Thus heterogeneous markups change the cutoff schedule, active mass, aggregate TFP, and the markup wedge, but not the Cobb--Douglas exponents. Independence of γ from (a, b) is the special case in which ψ_γ does not vary with γ.

6.15 Non-constant returns at the individual firm level

Suppose firm technology is

$$\widetilde{f}(a_i\ell_i, b_ik_i) = f(a_i\ell_i, b_ik_i)^\vartheta, \qquad 0 < \vartheta < 1,$$

where f is CRS. Let $c_i \equiv c(w/a_i, R/b_i)$ denote the unit-cost index of the inner technology. Producing y_i units requires $y_i^{1/\vartheta}$ units of the inner activity, so variable cost is

$$C_i(y_i) = c_i y_i^{1/\vartheta}.$$

Firm problem. Under CES final demand, write $\rho \equiv (\nu - 1)/\nu$. The firm solves

$$\max_{y_i \geq 0} \Big\{ PY^{1/\nu} y_i^\rho - c_i y_i^{1/\vartheta} - PFY \Big\}.$$

The first-order condition implies

$$y_i^* = \left(\vartheta\rho\frac{PY^{1/\nu}}{c_i}\right)^{1/(1/\vartheta-\rho)}.$$

Hence output, revenue, and profits are monotone powers of the same cost index c_i, so the active set remains a cutoff set in c_i. Substituting back into profits yields

$$c_i \le c_0(P, Y) = \zeta_{\nu,\vartheta,F} P\, Y^{1-1/\vartheta},$$

for a constant $\zeta_{\nu,\vartheta,F} > 0$. The cutoff is still one-dimensional, but unlike the CRS baseline it now depends on aggregate scale.

Balanced class. Now impose the same balanced interior class as in the baseline,

$$g(a, b) = \frac{1}{ab}\psi(a^\alpha b^\beta), \qquad m \equiv \alpha + \beta,$$

and normalize by the cutoff:

$$\tilde{a} \equiv a\frac{c_0}{w}, \qquad \tilde{b} \equiv b\frac{c_0}{R}.$$

Then the active set is fixed in $(\tilde{a}, \tilde{b})$ space and the transformed measure depends on factor prices only through

$$Q \equiv \frac{w^\alpha R^\beta}{c_0^m}.$$

If the associated scalar equation has an interior solution Q_*, then

$$c_0 = Q_*^{-1/m} w^{\alpha/m} R^{\beta/m}.$$

Combining this with the cutoff formula gives

$$P = B_\vartheta w^{\alpha/m} R^{\beta/m} Y^{(1-\vartheta)/\vartheta},$$

for a constant $B_\vartheta > 0$.

Aggregate production. At the firm level, variable factor payments are a constant fraction $\vartheta(\nu - 1)/\nu$ of revenue. On the balanced interior region, the same angular argument as in the baseline implies the split

$$wL = \vartheta\frac{\nu - 1}{\nu}\frac{\alpha}{m}PY, \qquad RK = \vartheta\frac{\nu - 1}{\nu}\frac{\beta}{m}PY.$$

Substituting these expressions into the inverse supply relation yields

$$Y = A_\vartheta L^{\vartheta\alpha/m} K^{\vartheta\beta/m},$$

for a constant $A_\vartheta > 0$. Therefore

$$\frac{wL}{RK} = \frac{\alpha}{\beta},$$

The macro elasticity of substitution remains one, and aggregate returns to scale fall to ϑ.

This is again an interior result. If support bounds begin to bind, the exact Cobb--Douglas form need not survive.

6.16 Fixed costs that are not proportional to output

Suppose each active firm pays a fixed operating cost FY^φ in units of the final good, where $0 \le \varphi \le 1$. This nests the baseline proportional case at $\varphi = 1$ and the additive case at $\varphi = 0$. Profits become

$$\pi_i = P\left\{\kappa_\nu \left(\frac{c(w/a_i, R/b_i)}{P}\right)^{1-\nu} Y - FY^\varphi\right\},$$

so the active set is still a cutoff set in unit cost,

$$\frac{c(w/a_i, R/b_i)}{P} \le \left(\kappa_\nu \frac{Y^{1-\varphi}}{F}\right)^{1/(\nu-1)}.$$

Thus the cutoff again moves with aggregate scale, and the balanced-class rescaling shows that the relevant scale variable is

$$\frac{F}{\left(L^{\frac{\alpha}{\alpha+\beta}} K^{\frac{\beta}{\alpha+\beta}}\right)^{1-\varphi}}.$$

Power-law interior. Now keep the balanced class

$$g(a, b) = \frac{1}{ab}\psi(a^\alpha b^\beta),$$

and specialize to the power-law subclass

$$\psi(z) = Dz^{-\tau}.$$

On the interior region where support bounds do not bind, the same angular argument as in the baseline implies

$$\frac{wL}{RK} = \frac{\alpha}{\beta}.$$

If

$$\frac{(2-\varphi)(\alpha+\beta)\tau + 1 - \varphi}{(\alpha+\beta)\tau + 1 - \varphi} < \nu < (\alpha+\beta)\tau + 1,$$

then for fixed F, gross and net output satisfy

$$Y \propto L^{\frac{\alpha}{\alpha+\beta}p(\varphi)} K^{\frac{\beta}{\alpha+\beta}p(\varphi)}, \qquad C \propto L^{\frac{\alpha}{\alpha+\beta}p(\varphi)} K^{\frac{\beta}{\alpha+\beta}p(\varphi)},$$

where

$$p(\varphi) = \frac{(\alpha+\beta)\tau(\nu-1)}{(\alpha+\beta)\tau(\nu-1) - (1-\varphi)\left((\alpha+\beta)\tau + 1 - \nu\right)}.$$

Thus fixed costs that are less than proportional to output preserve the unitary elasticity between capital and labor and the relative factor returns $wL/(RK) = \alpha/\beta$, but they raise aggregate returns to scale from one to $p(\varphi)$. In particular, $p(\varphi) > 1$ for $\varphi < 1$, while $p(1) = 1$.

Outside the power-law subclass, the balanced-class result remains an implicit one-dimensional transform of the scale variable above rather than a closed-form monomial. This is again an interior result. If support bounds begin to bind, the exact aggregate object need not remain a pure Cobb--Douglas monomial; in that case, it retains residual dependence on L/K.

6.17 CES micro technologies and aggregate CES substitution

Proposition 14. *Fix an aggregate elasticity $\varepsilon \in (0,1)$ and a firm-level elasticity $\varepsilon_f \in [0,\varepsilon]$. Let individual unit costs be*

$$c(a,b;w,R) = \left[\left(\frac{w}{a}\right)^{1-\varepsilon_f} + \left(\frac{R}{b}\right)^{1-\varepsilon_f}\right]^{1/(1-\varepsilon_f)}.$$

On an interior high-productivity region, write requirements as $(1/a, 1/b) = s(u, 1-u)$, with $s > 0$ and $u \in [0,1]$. Suppose the induced measure over requirements is cone-homogeneous,

$$d\mathcal{M}_R = s^{m-1}\, ds\, G(du), \qquad m > \nu - 1,$$

where G is finite and nonzero. Then there is a choice of G such that the aggregate unit cost satisfies

$$P(w,R) \propto \left(\lambda_L w^{1-\varepsilon} + \lambda_K R^{1-\varepsilon}\right)^{1/(1-\varepsilon)}$$

and relative factor demands satisfy

$$\frac{L}{K} = \frac{\lambda_L}{\lambda_K}\left(\frac{w}{R}\right)^{-\varepsilon}.$$

The result is exact on the interior region in which the active set is not truncated by the support bounds.

Proof. For type (s, u), unit cost is $s\, k_{\varepsilon_f}(u; w, R)$, where

$$k_{\varepsilon_f}(u; w, R) = \left[(wu)^{1-\varepsilon_f} + (R(1-u))^{1-\varepsilon_f}\right]^{1/(1-\varepsilon_f)}.$$

Radial integration over the zero-profit active set gives

$$P(w, R) \propto J_{\varepsilon_f,G}(w, R)^{-1/m}, \qquad J_{\varepsilon_f,G}(w, R) = \int_0^1 k_{\varepsilon_f}(u; w, R)^{-m}\, G(du).$$

The condition $m > \nu - 1$ is the radial integrability condition: the active-set integral contains a term of the form $\int_0^{\bar{s}(u)} s^{m-\nu}\, ds$. The markup and the fixed operating cost affect only constants and the mass of active firms, not the substitution shape.

Let $p = 1 - \varepsilon_f$, $\rho = 1 - \varepsilon$, and $\alpha = \rho/p$. Since $\varepsilon_f \leq \varepsilon$, $\alpha \in (0, 1]$. Let Z_L and Z_K be independent positive α-stable random variables with Laplace transforms

$$\mathrm{E}\left[e^{-tZ_L}\right] = e^{-\lambda_L t^\alpha}, \qquad \mathrm{E}\left[e^{-tZ_K}\right] = e^{-\lambda_K t^\alpha}.$$

Define G by requiring that, for every bounded measurable H,

$$\int_0^1 H\left(\left(\frac{1-u}{u}\right)^p\right) u^{-m}\, G(du) = \mathrm{E}\left[Z_L^{-m/p} H\left(\frac{Z_K}{Z_L}\right)\right].$$

Then

$$J_{\varepsilon_f,G}(w, R) = \mathrm{E}\left[(w^p Z_L + R^p Z_K)^{-m/p}\right].$$

If S is positive α-stable with $\mathrm{E}[e^{-tS}] = e^{-\Lambda t^\alpha}$, then for $q > 0$,

$$\mathrm{E}[S^{-q}] = \frac{1}{\Gamma(q)} \int_0^\infty x^{q-1} e^{-\Lambda x^\alpha}\, dx = \frac{\Gamma(q/\alpha)}{\alpha \Gamma(q)} \Lambda^{-q/\alpha}.$$

Applying this identity with $q = m/p$ and $\Lambda = \lambda_L w^{p\alpha} + \lambda_K R^{p\alpha}$ yields

$$J_{\varepsilon_f,G}(w, R) \propto \left(\lambda_L w^{1-\varepsilon} + \lambda_K R^{1-\varepsilon}\right)^{-m/(1-\varepsilon)}.$$

Therefore

$$P(w,R) \propto \left(\lambda_L w^{1-\varepsilon} + \lambda_K R^{1-\varepsilon}\right)^{1/(1-\varepsilon)}.$$

Shephard's lemma gives the displayed relative demand equation. $\square$

Two endpoint cases are useful for interpretation. If $\varepsilon_f = 0$ and $\varepsilon = 1/2$, the firm technology is Leontief and the stable law is Levy. When G has density h, the implied angular density reduces to

$$h(u) \propto \frac{u^{m-1/2}(1-u)^{m-1/2}}{\left(\lambda_L^2(1-u) + \lambda_K^2 u\right)^{m+1}},$$

which implies the productivity density used in the main text:

$$g(a,b) \propto \frac{a^{-1/2}b^{-1/2}}{\left(\lambda_L^2 a + \lambda_K^2 b\right)^{m+1}}.$$

If instead $\varepsilon_f = \varepsilon$, then $\alpha = 1$ and the stable mixture collapses to a point mass. This is a degenerate limiting member of the construction, not a case in which arbitrary angular heterogeneity preserves the CES index.

More generally, when G is absolutely continuous with density $h_{\varepsilon_f,\varepsilon}$, the corresponding productivity density can be written as

$$g_{\varepsilon_f,\varepsilon}(a,b) \propto (a+b)^{m-2} a^{-m} b^{-m} h_{\varepsilon_f,\varepsilon}\left(\frac{b}{a+b}\right).$$

The qualification is important: the construction defines a finite angular measure G, but it need not have a smooth density in every endpoint case.

6.18 Additive fixed costs and non-CRS aggregation

The baseline market model uses a per-firm operating cost equal to FY final-good units, so firm profits are proportional to aggregate output Y and the aggregate object is constant returns. Here I replace that with a *constant* operating cost $F > 0$ per active firm and characterize what survives of the Cobb-Douglas aggregation logic.

Let $m \equiv \alpha + \beta$ and

$$Q(L,K) \equiv L^{\alpha/m} K^{\beta/m}.$$

For the balanced class

$$g(a,b) = \frac{1}{ab}\psi(a^\alpha b^\beta), \qquad a \geq a_{\min},\ b \geq b_{\min},$$

there is an exact market-side rescaling, but globally it is *two-dimensional*:

$$Y(L,K;F) = Q(L,K)\,\mathcal{Y}\!\left(\frac{L}{K}, \frac{F}{Q(L,K)}\right), \qquad C(L,K;F) = Q(L,K)\,\mathcal{C}\!\left(\frac{L}{K}, \frac{F}{Q(L,K)}\right),$$

where $C = Y - FM$ is net final output and M is the mass of active firms. Hence with positive lower support bounds there is generally no globally exact representation as a function of $Q(L,K)$ alone. The residual dependence on L/K is the same support-bound effect that appears in the baseline rescaling.

On the *homogeneous interior* of the active set, where those support bounds are slack, the ratio dependence disappears for the entire balanced class:

$$Y(L,K;F) = Q(L,K)\,\mathcal{Y}\!\left(\frac{F}{Q(L,K)}\right), \qquad C(L,K;F) = Q(L,K)\,\mathcal{C}\!\left(\frac{F}{Q(L,K)}\right).$$

So for fixed F the market economy does collapse to a one-dimensional function of the Cobb-Douglas composite on the interior region. Moreover, on that same region,

$$\frac{w}{R} = \frac{\alpha}{\beta}\frac{K}{L},$$

so the unit-elastic ratio margin survives. What fails is constant returns: additive fixed costs introduce the extra scale variable F/Q.

For the smaller power-law subclass $\psi(z) = Dz^{-\tau}$ (independent Pareto is $\tau = 1$), the interior object becomes an explicit power of Q:

$$Y = A_Y F^{1-p} Q^p, \qquad C = A_C F^{1-p} Q^p, \qquad p = \frac{m\tau(\nu - 1)}{\nu(m\tau + 1) - (2m\tau + 1)},$$

under the regularity condition

$$\frac{2m\tau + 1}{m\tau + 1} < \nu < m\tau + 1.$$

Thus the market analogue is not globally CRS Cobb-Douglas; rather, the balanced class gives a *globally exact two-state decomposition*, an *interior one-state reduction*, and only the power-law subclass gives an explicit homogeneous power.

6.18.1 Market setup with additive fixed costs

Let $\mu \equiv \nu/(\nu - 1)$ denote the constant markup. A firm of type (a,b) has unit variable cost $c(w/a, R/b)$ and labor share $\eta(w/a, R/b)$. With additive fixed cost F final-good units, profits are

$$\pi(a,b) = P\left\{\kappa_\nu \left(\frac{c(w/a, R/b)}{P}\right)^{1-\nu} Y - F\right\}. \tag{45}$$

Hence the firm is active if and only if

$$\frac{c(w/a, R/b)}{P} \le \zeta, \qquad \zeta \equiv \left(\kappa_\nu \frac{Y}{F}\right)^{1/(\nu-1)}. \tag{46}$$

Relative to the proportional-cost benchmark, the key difference is that the cutoff ζ is now endogenous and increases with gross output. The active mass is

$$M = \int_\Omega g(a,b)\,\mathrm{d}a\,\mathrm{d}b,$$

and net final output is

$$C = Y - FM. \tag{47}$$

The standard market equations become

$$P^{1-\nu} = \mu^{1-\nu} \int_\Omega c(w/a, R/b)^{1-\nu} g(a,b)\,\mathrm{d}a\,\mathrm{d}b, \tag{48}$$

$$\frac{wL}{PY} = \mu^{-\nu} \int_\Omega \eta(w/a, R/b) \left(\frac{c(w/a, R/b)}{P}\right)^{1-\nu} g(a,b)\,\mathrm{d}a\,\mathrm{d}b, \tag{49}$$

$$\frac{RK}{PY} = \mu^{-\nu} \int_\Omega (1 - \eta(w/a, R/b)) \left(\frac{c(w/a, R/b)}{P}\right)^{1-\nu} g(a,b)\,\mathrm{d}a\,\mathrm{d}b. \tag{50}$$

Adding (49) and (50) and using (48) gives the familiar constant variable-cost share,

$$\frac{wL + RK}{PY} = \mu^{-1} = \frac{\nu - 1}{\nu}. \tag{51}$$

Thus the markups themselves are not the issue here; the new object is the endogenous cutoff (46).

6.18.2 Exact rescaling for the balanced class

Set

$$s \equiv \frac{L}{K}, \qquad Q(L,K) \equiv L^{\alpha/m} K^{\beta/m}, \qquad \phi \equiv \frac{F}{Q(L,K)}, \qquad m \equiv \alpha + \beta.$$

Write the real factor prices as

$$\omega \equiv \frac{w}{P}, \qquad \varrho \equiv \frac{R}{P}, \qquad \widehat{Y} \equiv \frac{Y}{Q(L,K)}.$$

Now transform productivities and real factor prices by

$$\widehat{a} = a\, s^{\beta/m}, \qquad \widehat{b} = b\, s^{-\alpha/m}, \qquad \widehat{\omega} = \omega\, s^{\beta/m}, \qquad \widehat{\varrho} = \varrho\, s^{-\alpha/m}. \tag{52}$$

Because $g(a,b) = a^{-1}b^{-1}\psi(a^{\alpha}b^{\beta})$, one has the exact measure invariance

$$g(a,b)\,\mathrm{d}a\,\mathrm{d}b = g(\widehat{a},\widehat{b})\,\mathrm{d}\widehat{a}\,\mathrm{d}\widehat{b}. \tag{53}$$

Moreover,

$$\frac{c(w/a, R/b)}{P} = c(\omega/a, \varrho/b) = c(\widehat{\omega}/\widehat{a}, \widehat{\varrho}/\widehat{b}),$$

and the support becomes

$$\widehat{a} \geq a_{\min} s^{\beta/m}, \qquad \widehat{b} \geq b_{\min} s^{-\alpha/m}. \tag{54}$$

Finally, since

$$L = Q(L,K) s^{\beta/m}, \qquad K = Q(L,K) s^{-\alpha/m},$$

we have

$$\frac{\omega L}{Y} = \frac{\widehat{\omega}}{\widehat{Y}}, \qquad \frac{\varrho K}{Y} = \frac{\widehat{\varrho}}{\widehat{Y}}.$$

Therefore the equilibrium system becomes

$$\begin{aligned}
1 &= \mu^{1-\nu} \int_{\widehat{\Omega}} c(\widehat{\omega}/\widehat{a}, \widehat{\varrho}/\widehat{b})^{1-\nu} g(\widehat{a},\widehat{b})\,\mathrm{d}\widehat{a}\,\mathrm{d}\widehat{b}, && (55)\\
\frac{\widehat{\omega}}{\widehat{Y}} &= \mu^{-\nu} \int_{\widehat{\Omega}} \eta(\widehat{\omega}/\widehat{a}, \widehat{\varrho}/\widehat{b}) c(\widehat{\omega}/\widehat{a}, \widehat{\varrho}/\widehat{b})^{1-\nu} g(\widehat{a},\widehat{b})\,\mathrm{d}\widehat{a}\,\mathrm{d}\widehat{b}, && (56)\\
\frac{\widehat{\varrho}}{\widehat{Y}} &= \mu^{-\nu} \int_{\widehat{\Omega}} \big(1 - \eta(\widehat{\omega}/\widehat{a}, \widehat{\varrho}/\widehat{b})\big) c(\widehat{\omega}/\widehat{a}, \widehat{\varrho}/\widehat{b})^{1-\nu} g(\widehat{a},\widehat{b})\,\mathrm{d}\widehat{a}\,\mathrm{d}\widehat{b}, && (57)\\
\widehat{\Omega} &= \Big\{ (\widehat{a},\widehat{b}) : c(\widehat{\omega}/\widehat{a}, \widehat{\varrho}/\widehat{b}) \leq \zeta,\ \widehat{a} \geq a_{\min} s^{\beta/m},\ \widehat{b} \geq b_{\min} s^{-\alpha/m} \Big\}, && (58)\\
\zeta^{\nu-1} &= \kappa_{\nu} \frac{\widehat{Y}}{\phi}. && (59)
\end{aligned}$$

Proposition 1 (Exact global decomposition for the balanced class)**.** *For the balanced class, the normalized equilibrium system* (55)–(59) *depends on* (L, K, F) *only through*

$$s = \frac{L}{K}, \qquad \phi = \frac{F}{Q(L,K)}.$$

Consequently, whenever the modified equilibrium is unique, there exist functions $\mathcal{Y}$*,* $\mathcal{C}$*, and*

$\mathcal{M}$ *such that*

$$Y(L,K;F) = Q(L,K)\,\mathcal{Y}\!\left(\frac{L}{K},\frac{F}{Q(L,K)}\right), \tag{60}$$

$$C(L,K;F) = Q(L,K)\,\mathcal{C}\!\left(\frac{L}{K},\frac{F}{Q(L,K)}\right), \tag{61}$$

with

$$\mathcal{C}(s,\phi) = \mathcal{Y}(s,\phi) - \phi\,\mathcal{M}(s,\phi).$$

In particular, with positive lower support bounds the additive-cost economy is generally not *globally a function of $Q(L,K)$ alone.*

This is the market analogue of the planner-side exact rescaling. The additional normalized state variable is F/Q, and the residual dependence on L/K comes entirely from the shifted lower bounds (54).

Remark 1 (What survives beyond independent Pareto)*.* The exact decomposition (60) is not special to the independent Pareto benchmark. It holds for the whole balanced class because the transformation (52) leaves the measure invariant. Outside that class, the same argument no longer goes through, so the appendix restricts this extension to the balanced family.

6.18.3 Interior region: one-dimensional reduction and unit elasticity

Now impose the same interior restriction: after the balanced rescaling, the equilibrium active set is not clipped by the lower support bounds in (54). On that region the active set is determined only by the cost cutoff.

Set

$$x \equiv \frac{\widehat{\omega}}{\widehat{a}}, \qquad y \equiv \frac{\widehat{\varrho}}{\widehat{b}}, \qquad q \equiv \frac{x}{y}, \qquad u_c \equiv c(x,y), \qquad H(q) \equiv c(q,1).$$

Then, by homogeneity of the unit-cost function,

$$y = \frac{\nu_c}{H(q)}, \qquad x = \frac{q\nu_c}{H(q)}.$$

Define further

$$A(q) \equiv q^{-\alpha}H(q)^m, \qquad \eta(q) \equiv \frac{qH'(q)}{H(q)}. \tag{62}$$

Using the same change of variables as above, the balanced density induces the kernel

$$g(a,b)\,\mathrm{d}a\,\mathrm{d}b = \frac{1}{q\nu_c}\,\psi\big(XA(q)\nu_c^{-m}\big)\,\mathrm{d}q\,\mathrm{d}\nu_c, \qquad X \equiv \widehat{\omega}^{\alpha}\widehat{\varrho}^{\beta} = \left(\frac{w}{P}\right)^{\alpha}\left(\frac{R}{P}\right)^{\beta}. \tag{63}$$

The interior active set is simply $0 < \nu_c \leq \zeta$.

Hence the price index becomes

$$1 = \mu^{1-\nu}\Phi(X,\zeta), \qquad \Phi(X,\zeta) \equiv \int_0^{\zeta}\int_0^{\infty} q^{-1}\nu_c^{-\nu}\psi\big(XA(q)\nu_c^{-m}\big)\,\mathrm{d}q\,\mathrm{d}\nu_c. \tag{64}$$

When the normalized price-index equation is single-valued, write the implied solution as $X = X(\zeta)$.

The next lemma is the key market-side version of the balanced-class logic used above.

Lemma 1 (Conditional mean labor share on the balanced interior)**.** *On the homogeneous interior of the active set,*

$$\mathbb{E}[\eta \mid \nu_c] = \frac{\alpha}{m} \qquad \textit{for every interior cost level } \nu_c. \tag{65}$$

Proof. From (62),

$$\frac{\mathrm{d}}{\mathrm{d}q}\log A(q) = \frac{m\eta(q) - \alpha}{q}. \tag{66}$$

Holding ν_c fixed, the conditional density of q is proportional to

$$q^{-1}\psi\big(XA(q)\nu_c^{-m}\big)\,.$$

Therefore

$$m\mathbb{E}[\eta \mid \nu_c] - \alpha = \frac{\int_0^{\infty}\psi(XA(q)\nu_c^{-m})\,\mathrm{d}\log A(q)}{\int_0^{\infty} q^{-1}\psi(XA(q)\nu_c^{-m})\,\mathrm{d}q}.$$

On the homogeneous interior, $A(q) \to \infty$ as $q \downarrow 0$ and as $q \uparrow \infty$, and $A(q)$ has a unique minimum. Hence the two monotone branches cancel after the change of variables $z = A(q)$, so the numerator is zero. This yields (65). □

Proposition 2 (Interior factor shares and unit elasticity)**.** *On the homogeneous interior of the balanced class,*

$$\frac{wL}{PY} = \frac{\alpha}{m}\frac{\nu-1}{\nu}, \qquad \frac{RK}{PY} = \frac{\beta}{m}\frac{\nu-1}{\nu}. \tag{67}$$

Therefore

$$\frac{w}{R} = \frac{\alpha}{\beta}\frac{K}{L}. \tag{68}$$

So the market economy preserves unit elasticity in the ratio margin throughout the balanced interior class.

Proof. Using (63), labor-market clearing can be written as

$$\frac{wL}{PY} = \mu^{-\nu}\int_0^{\zeta}\int_0^{\infty} \eta(q)q^{-1}\nu_c^{-\nu}\psi\big(XA(q)\nu_c^{-m}\big)\,\mathrm{d}q\,\mathrm{d}\nu_c.$$

Condition first on ν_c and apply Lemma 1. This gives

$$\frac{wL}{PY} = \frac{\alpha}{m}\mu^{-\nu}\Phi(X,\zeta).$$

Now use (64): $\mu^{1-\nu}\Phi(X,\zeta) = 1$, hence $\mu^{-\nu}\Phi(X,\zeta) = \mu^{-1} = (\nu-1)/\nu$. This yields the first formula in (67); the second follows symmetrically, and (68) then follows from division. $\square$

Proposition 3 (Interior one-dimensional reduction for the balanced class)**.** *Assume the support bounds are slack and the normalized price-index equation* (64) *is single-valued, so write* $X = X(\zeta)$*. Then gross output satisfies*

$$Y = B\,X(\zeta)^{1/m}Q(L,K), \qquad B \equiv \mu\left(\frac{m}{\alpha}\right)^{\alpha/m}\left(\frac{m}{\beta}\right)^{\beta/m}. \tag{69}$$

The cutoff is determined by

$$\phi = \kappa_\nu B X(\zeta)^{1/m}\zeta^{1-\nu}, \qquad \phi \equiv \frac{F}{Q(L,K)}. \tag{70}$$

Hence, on the homogeneous interior, there exist one-dimensional functions $\mathcal{Y}$ *and* $\mathcal{C}$ *such that*

$$Y(L,K;F) = Q(L,K)\,\mathcal{Y}\!\left(\frac{F}{Q(L,K)}\right), \tag{71}$$

$$C(L,K;F) = Q(L,K)\,\mathcal{C}\!\left(\frac{F}{Q(L,K)}\right). \tag{72}$$

Equivalently, for fixed F*, the market object collapses to a function of* $Q(L,K)$ *alone:*

$$C(L,K;F) = \mathfrak{F}_F\big(Q(L,K)\big).$$

Proof. From Proposition 2,

$$\frac{w}{P} = \frac{\alpha}{m}\frac{\nu-1}{\nu}\frac{Y}{L}, \qquad \frac{R}{P} = \frac{\beta}{m}\frac{\nu-1}{\nu}\frac{Y}{K}.$$

Therefore

$$X = \left(\frac{w}{P}\right)^{\alpha}\left(\frac{R}{P}\right)^{\beta} = \left(\frac{\nu-1}{\nu}\right)^{m}\left(\frac{\alpha}{m}\right)^{\alpha}\left(\frac{\beta}{m}\right)^{\beta}\left(\frac{Y}{Q(L,K)}\right)^{m},$$

which is equivalent to (69). Substituting (69) into the cutoff formula (46) yields (70). Since, on the interior, both X and the active mass depend on the equilibrium only through ζ, the right-hand sides of (71) and (72) are functions of $\phi = F/Q$ alone. $\square$

Remark 2 (What this says and what it does not say)*.* Proposition 3 summarizes the cleanest market result in the additive-cost case.

It says that on the homogeneous interior the additive-cost market economy retains the Cobb-Douglas *ratio margin* of the original paper, but it loses constant returns because scale enters through F/Q. In the general balanced class this one-dimensional object is implicit, not a power law. So the correct statement is not "the original Cobb-Douglas survives," but rather "the market object is still a one-dimensional transform of the Cobb-Douglas composite on the interior region."

6.18.4 Power-law subclass: explicit non-CRS Cobb-Douglas power

Now specialize to

$$\psi(z) = Dz^{-\tau}, \qquad \tau > 0,$$

which nests independent Pareto as $\tau = 1$. On the homogeneous interior, (64) factorizes:

$$\Phi(X, \zeta) = B_P X^{-\tau} \zeta^{m\tau+1-\nu}, \tag{73}$$

for some constant $B_P > 0$ that depends only on primitives and the micro cost function. The active mass similarly satisfies

$$M(X, \zeta) = B_M X^{-\tau} \zeta^{m\tau}, \tag{74}$$

for some $B_M > 0$.

As long as the price-index integral is finite,

$$\nu < m\tau + 1, \tag{75}$$

so that $m\tau + 1 - \nu > 0$. Then (73) implies

$$X(\zeta) = X_0 \zeta^{(m\tau+1-\nu)/\tau} \tag{76}$$

for some constant $X_0 > 0$. Substituting into (69) gives

$$Y = A_0 Q \zeta^{\theta}, \qquad \theta \equiv \frac{m\tau + 1 - \nu}{m\tau}, \tag{77}$$

for some constant $A_0 > 0$.

Moreover, combining (74) with (76) yields

$$M(\zeta) = \widetilde{B}_M \zeta^{\nu-1} \tag{78}$$

for some $\widetilde{B}_M > 0$. Hence the fixed-cost share in gross output is constant on the interior power-law solution:

$$\frac{FM}{Y} = \kappa_\nu \widetilde{B}_M. \tag{79}$$

To obtain a well-behaved interior expanding-cutoff solution, one further wants

$$\theta < \nu - 1 \qquad \Longleftrightarrow \qquad \nu > \frac{2m\tau + 1}{m\tau + 1}. \tag{80}$$

Under (75) and (80), define

$$e \equiv \nu - 1 - \theta = \frac{\nu(m\tau + 1) - (2m\tau + 1)}{m\tau} > 0. \tag{81}$$

Then (46) and (77) imply

$$\zeta = \widetilde{A}_\zeta \left(\frac{Q}{F}\right)^{1/e}$$

for some $\widetilde{A}_\zeta > 0$, and therefore

$$Y = A_Y F^{1-p} Q^p, \tag{82}$$

$$C = A_C F^{1-p} Q^p, \tag{83}$$

with positive constants A_Y, A_C and exponent

$$p = 1 + \frac{\theta}{e} = \frac{m\tau(\nu - 1)}{\nu(m\tau + 1) - (2m\tau + 1)}. \tag{84}$$

Since $e > 0$, one has $p > 1$.

Proposition 4 (Explicit power-law market formula)**.** *Suppose $\psi(z) = Dz^{-\tau}$ and the equilibrium remains on the homogeneous interior. If*

$$\frac{2m\tau + 1}{m\tau + 1} < \nu < m\tau + 1,$$

then the additive-cost market economy admits the explicit non-CRS Cobb-Douglas-power representation

$$Y = A_Y F^{1-p} Q(L, K)^p, \qquad C = A_C F^{1-p} Q(L, K)^p,$$

with exponent p given by (84)*. Independent Pareto is the special case $\tau = 1$.*

Remark 3 (Interpretation of the power-law formula). This is the decentralized-market analogue of the planner's interior power law. The elasticity of substitution in the labor-capital ratio is still one, because the factor-price relation remains (68). But returns to scale are no longer constant: on the interior power-law solution, the market economy is homogeneous of degree $p > 1$ in Q, not degree one.

Remark 4 (If the power-law regularity fails). If $\nu \geq m\tau + 1$, the price-index integral itself is not finite on the interior. If instead

$$\nu \leq \frac{2m\tau + 1}{m\tau + 1},$$

then the slack-support power formula implies a non-expanding or perverse cutoff response to scale. In that region the simple interior power law is not a complete equilibrium characterization; one should instead expect support clipping to matter, and the correct object is the global two-state decomposition of Proposition 1.